

# Ideal-gas-like market models with savings: Quenched and annealed cases

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## Abstract

We analyze the ideal-gas-like models of markets and review the different cases where a ‘savings’ factor changes the nature and shape of the distribution of wealth. These models can produce similar distribution of wealth as observed across varied economies. We present a more realistic model where the saving factor can vary over time (annealed savings) and yet produce Pareto distribution of wealth in certain cases. We discuss the relevance of such models in the context of wealth distribution, and address some recent issues in the context of these models.

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## 1. Introduction

The study of wealth distribution [1] in a society has remained an interesting problem since Pareto’s first observation [2] that the number of rich people decay with wealth  $m$  following an inverse law:

$$P(m) \sim m^{-(1+\nu)}. \quad (1)$$

Here,  $P(m)$  is number density of people possessing wealth  $m$ . The exponent  $\nu$  is known to be the Pareto exponent, and this generally assumes a value between 1 and 3 in varied economies [3–5]. It is also known that for low and medium income, the number density  $P(m)$  falls off much faster: exponentially [3] or in a log-normal way [4].

In recent years, easy availability of data has helped in the analysis of wealth or income distributions in various societies [1]. It is now more or less established that the distribution has a power-law tail for the large (about 5% of the population) wealth/income [3–5], while the majority (around 95%) low-income distribution fits well to Gibbs or log-normal form.

There has been several attempts to model a simple economy with minimum trading ingredients, which involve a wealth exchange process [6] that produce a distribution of wealth similar to that observed in the real market. We are particularly interested in microscopic models of markets where the (economic) trading activity

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is considered as a scattering process [7–14] (see also Ref. [15] for a recent extensive review). We concentrate on models that incorporate ‘savings’ as an essential ingredient in a trading process, and reproduces the salient features seen across wealth distributions in varied economies (see Ref. [16] for a review). Angle [17] studied inequality processes which can be mapped to the savings wealth models in certain cases; see Ref. [18] for a detailed review.

These studies also show (and discussed briefly here) how the distribution of savings can be modified to reproduce the salient features of empirical distributions of wealth—namely the shape of the distribution for the low and middle wealth and the tunable Pareto exponent. In all these models [9–13], ‘savings’ was introduced as an annealed parameter that remained invariant with time (or trading).

Apart from a brief summary of the established results in such models, here we report some new results for annealed cases, where the saving factor can change with time, as one would expect in a real trading process. We report cases where the wealth distribution is still described by a Pareto law. We also forward some justification of the various assumptions in such models.

## 2. Ideal-gas-like models of trading

We first consider an ideal-gas model of a closed economic system. Wealth is measured in terms of the amount of money possessed by an individual. In the following models, money is the only quantity involved in the trading processes, hence, the terms ‘money’ and ‘wealth’ would be used to mean the same quantity. Production or destruction are not allowed, i.e., total money  $M$  is fixed, and also there is no migration of population, i.e., total number of agents  $N$  is fixed, and the only economic activity is confined to trading. A detailed discussion on the conservation of money and related issues is given in Section 5.1. Each agent  $i$ , individual or corporate, possesses money  $m_i(t)$  at time  $t$ . In any trading, a pair of agents  $i$  and  $j$  randomly exchange their money [7–9] such that their total money is (locally) conserved and none end up with negative money ( $m_i(t) \geq 0$ , i.e., debt not allowed):

$$m_i(t) + m_j(t) = m_i(t+1) + m_j(t+1). \quad (2)$$

Time ( $t$ ) changes by one unit after each trading. The steady-state ( $t \rightarrow \infty$ ) distribution of money is Gibbs one:

$$P(m) = (1/T) \exp(-m/T), \quad T = M/N. \quad (3)$$

No matter how uniform or justified the initial distribution is, the eventual steady state corresponds to Gibbs distribution where most of the people end up with very little money. This follows from the conservation of money and additivity of entropy:

$$P(m_1)P(m_2) = P(m_1 + m_2). \quad (4)$$

This steady-state result is quite robust and realistic. Several variations of the trading, and of the ‘lattice’ (on which the agents can be put and each agent trades with its ‘lattice neighbors’ only)—compact, fractal or small-world like [1], does not affect the distribution.

## 3. Savings in ideal-gas trading market: quenched case

In any trading, savings come naturally [19]. A saving factor  $\lambda$  is therefore introduced in the same model [9] (Ref. [8] is the model without savings), where each trader at time  $t$  saves a fraction  $\lambda$  of its money  $m_i(t)$  and trades randomly with the rest. In each of the following two cases, the savings fraction does not vary with time, and hence we call it ‘quenched’ in the terminology of statistical mechanics.

### 3.1. Fixed or uniform savings

For the case of ‘fixed’ savings, the money-exchange rules are

$$m_i(t+1) = m_i(t) + \Delta m, \quad m_j(t+1) = m_j(t) - \Delta m, \quad (5)$$

where

$$\Delta m = (1 - \lambda)[\varepsilon\{m_i(t) + m_j(t)\} - m_i(t)], \quad (6)$$

where  $\varepsilon$  is a random fraction, coming from the stochastic nature of the trading.  $\lambda$  is a fraction ( $0 \leq \lambda < 1$ ) which we call the saving factor.

The market (non-interacting at  $\lambda = 0$  and 1) becomes ‘interacting’ for any non-vanishing  $\lambda (< 1)$ : for fixed  $\lambda$  (same for all agents), the steady-state distribution  $P_f(m)$  of money is sharply decaying on both sides with the most-probable money per agent shifting away from  $m = 0$  (for  $\lambda = 0$ ) to  $M/N$  as  $\lambda \rightarrow 1$ . The self-organizing feature of this market, induced by sheer self-interest of saving by each agent without any global perspective, is very significant as the fraction of paupers decrease with saving fraction  $\lambda$  and most people possess some fraction of the average money in the market (for  $\lambda \rightarrow 1$ , the socialists’ dream is achieved with just people’s self-interest of saving!). Although this fixed saving propensity does not give the Pareto-like power-law distribution yet, the Markovian nature of the scattering or trading processes (Eq. (4)) is lost and the system becomes co-operative. Indirectly through  $\lambda$ , the agents get to develop a correlation with (start interacting with) each other and the system co-operatively self-organizes [20] towards a most-probable distribution.

This model has been understood to a certain extent (see, e.g., Refs. [21–23]), and argued to resemble a gamma distribution [22], and partly explained analytically. This model clearly finds its relevance in cases where the economy consists of traders with ‘waged’ income [24].

### 3.2. Distributed savings

In a real society or economy,  $\lambda$  is a very inhomogeneous parameter: the interest of saving varies from person to person. We move a step closer to the real situation where saving factor  $\lambda$  is widely distributed within the population [11–13]. The evolution of money in such a trading can be written as

$$m_i(t+1) = \lambda_i m_i(t) + \varepsilon_{ij}[(1 - \lambda_i)m_i(t) + (1 - \lambda_j)m_j(t)], \quad (7)$$

$$m_j(t+1) = \lambda_j m_j(t) + (1 - \varepsilon_{ij})[(1 - \lambda_i)m_i(t) + (1 - \lambda_j)m_j(t)]. \quad (8)$$

One again follows the same rules as before, except that

$$\Delta m = \varepsilon_{ij}(1 - \lambda_j)m_j(t) - (1 - \lambda_i)(1 - \varepsilon_{ij})m_i(t). \quad (9)$$

Here,  $\lambda_i$  and  $\lambda_j$  are the saving propensities of agents  $i$  and  $j$ . The agents have fixed (over time) saving propensities, distributed independently, randomly and uniformly (white) within an interval 0 to 1, and agent  $i$  saves a random fraction  $\lambda_i$  ( $0 \leq \lambda_i < 1$ ) and this  $\lambda_i$  value is quenched for each agent ( $\lambda_i$  are independent of trading or  $t$ ).  $P(m)$  is found to follow a strict power-law decay. This decay fits to Pareto law (1) with  $\nu = 1.01 \pm 0.02$  for several decades. This power law is extremely robust: a distribution

$$\rho(\lambda) \sim |\lambda_0 - \lambda|^\alpha, \quad \lambda_0 \neq 1, \quad 0 \leq \lambda < 1, \quad (10)$$

of quenched  $\lambda$  values among the agents produce power-law distributed  $m$  with Pareto index  $\nu = 1$ , irrespective of the value of  $\alpha$ . For negative  $\alpha$  values, however, we get an initial (small  $m$ ) Gibbs-like decay in  $P(m)$ . In case  $\lambda_0 = 1$ , the Pareto exponent is modified to  $\nu = 1 + \alpha$ , which qualifies for the non-universal exponents in the same model [11,25].

This model [11] has been thoroughly analyzed, and the analytical derivation of the Pareto exponent has been achieved in certain cases [23,25,26]. The Pareto exponent has been derived to exactly 1.

In this model, agents with higher saving propensity tend to hold higher average wealth, which is justified by the fact that the saving propensity in the rich population is always high [27].

## 4. Savings in Ideal-gas trading market: annealed case

In a real trading process, the concept of ‘saving factor’ cannot be attributed to a quantity that is invariant with time. A saving factor always changes with time or trading. In earlier works, we reported the case of annealed savings, where the savings factor  $\lambda_i$  changes with time in the interval  $[0, 1)$ , but does not produce a

power law in  $P(m)$  [11]. We report below some special cases of annealed saving which produce a power-law distribution of  $P(m)$ .

If one allows the saving factor  $\lambda_i$  to vary with time in  $[0, 1)$ , the money distribution  $P(m)$  does not produce a power-law tail.

Instead, we propose a slightly different model of an annealed saving case. We associate a parameter  $\mu_i$  ( $0 < \mu_i < 1$ ) with each agent  $i$  such that the savings factor  $\lambda_i$  randomly assumes a value in the interval  $[\mu_i, 1)$  at each time or trading. The trading rules are of course unaltered and governed by Eqs. (7) and (8). Now, considering a suitable distribution  $\zeta(\mu)$  of  $\mu$  over the agents, one can produce money distributions with power-law tail. The only condition that needs to hold is that  $\zeta(\mu)$  should be non-vanishing as  $\mu \rightarrow 1$ . Fig. 1 shows the case when  $\zeta(\mu) = 1$ . Numerical simulations suggest that the behavior of the wealth distribution is similar to the quenched savings case. In other words, only if  $\zeta(\mu) \propto |1 - \mu|^\alpha$ , it is reflected in the Pareto exponent as  $\nu = 1 + \alpha$ .

## 5. Relevance of gas-like models

All these gas-like models of trading markets are based on the assumption of (a) money conservation (globally in the market; as well as locally in any trading) and (b) stochasticity. These points have been criticized strongly (by economists) in the recent literature [28]. In the following, we forward some of the arguments in favor of these assumptions (see also Ref. [29]).

### 5.1. Money conservation

If we view the trading as scattering processes, one can see the equivalence. Of course, in any such ‘money-exchange’ trading process, one receives some profit or service from the other and this does not appear to be completely random, as assumed in the models. However, if we concentrate only on the ‘cash’ exchanged (even using bank cards!), every trading is a money conserving one (like the elastic scattering process in physics!).

In another recent study [30], we investigated the behavior of the steady-state distributions of money, commodity and total wealth, with local (in time) fluctuations in the price of the commodity over the ‘just price’. These distributions studied in markets with agents having uniform and random saving factors show self-organizing features in money distribution similar to the cases without any commodity (or with consumable commodities), while the commodity distribution shows an exponential decay. The wealth distribution shows interesting behavior: gamma like distribution for uniform saving propensity and has the

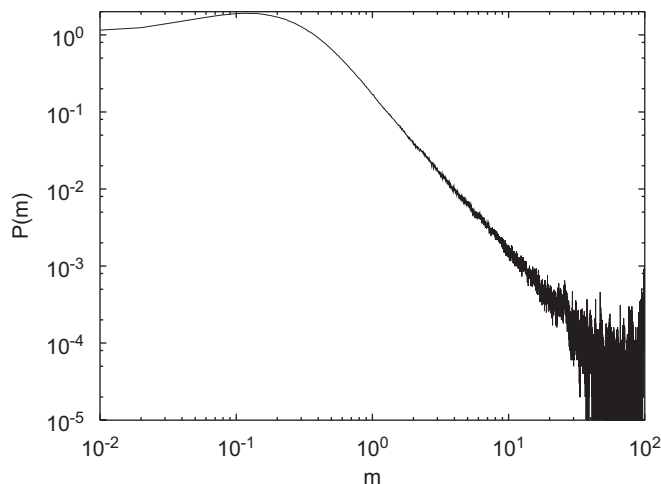


Fig. 1. Distribution  $P(m)$  of money  $m$  in case of annealed savings  $\lambda$  varying randomly in  $[\mu, 1)$ . Here,  $\zeta(\mu)$  has a uniform distribution. The distribution has a power-law tail with Pareto index  $\nu = 1$ . The simulation has been done for a system of  $N = 10^2$  agents, with  $M/N = 1$ .  $P(m)$  is the steady-state distribution after  $4 \times 10^4 \times N$  random exchanges, and averaged over an ensemble of  $10^5$ .

same power-law tail, as that of the money distribution for a market with agents having random saving propensity. This effectively accounts for some relaxation over the strict money or wealth conservation at each trading; the overall distribution, however, remains the same [30] (with identical Pareto exponent).

It is also important to note that the frequency of money exchange in such models defines a time scale in which the total money in the market does not change. In real economies, the total money changes much slowly, so that in the time scale of exchanges, it is quite reasonable to assume the total money to be conserved in these exchange models. This (not so strict/relaxed) conservation can also be justified by the fact that the average international foreign (currency) exchange rates do not change drastically (say, by more than 10%) very often. For example, according to the Reserve Bank of India, the US Dollar remained at  $\text{INR } 45 \pm 3$  for the last 8 years [31]! These exchange rates indeed reflect the relative (total) wealths of the two countries, who might even go for major loans/debts as part of their regular economic activities. Therefore, these stabilities of the foreign exchange rates over such large time scales clearly indicate the conservation of effective wealth over day-to-day trading.

## 5.2. Stochasticity

But, are these trading random? Surely not, when looked upon from individual's point of view: when one maximizes his/her utility by money exchange for the  $p$ th commodity, he/she may choose to go to the  $q$ th agent and for the  $r$ th commodity he/she will go to the  $s$ th agent. But since  $p \neq q \neq r \neq s$  in general, when viewed from a global level, these trading/scattering events will all look random (although for individuals these is a defined choice or utility maximization).

Apart from the choice of the agents for any trade, the traded amounts are considered to be random in such models. Some critics argue this cannot be totally random as the amount is determined by the price of the commodity exchanged. Again, this can be defended very easily. If a little fluctuation over the 'just price' occurs in each trade due to the bargain capacities of the agents involved, one can easily demonstrate that after sufficient trading (time, depending on the amount of fluctuation in each trade), the distribution will be determined by the stochasticity, as in the cases of directed random walks or other biased stochastic models in physics.

It may be noted in this context that in the stochastically formulated ideal-gas models in physics (developed in late 1800/early 1900) one (physicists) already knew for more than a hundred years that each of the constituent particle (molecule) follows a precise equation of motion, namely that due to Newton. The assumption of stochasticity here in such models, even though each agent might follow an utility maximizing strategy (like Newton's equation of motion for molecules), is therefore not very unusual in the context.

## 6. Summary and conclusions

We analyze the gas-like models of markets. We review the different cases where a quenched 'savings' factor changes the nature and shape of the distribution of wealth. Suitable modification in the nature of the 'savings' distribution can simulate all observed wealth distributions. We give here some new numerical results for the annealed 'savings' case. We find that the more realistic model, where the saving factor randomly varies in time (annealed savings), still produces a Pareto distribution of wealth in certain cases. These results indicate that small fluctuations in the saving pattern of individuals or groups do not affect the novel results, viz. the self-organization and emergence of power-law tail in money distribution in the gas-like models. We also forward some arguments in favor of the assumptions made in such gas-like models.

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