Available online at www.sciencedirect.com
ScienceDirect
Physica A 382 (2007) 149–171
PHYSICA A
www.elsevier.com/locate/physa

Self-consistent asset pricing models

Y. Malevergne[a,b], D. Sornette[a,*]

[a]Department of Management, Technology and Economics, ETH Zurich, Kreuzplatz, 5, CH-8032 Zurich, Switzerland
[b]EM-Lyon Graduate School of Management, 23 Avenue Guy de Collongue, 69134 Ecully Cedex, France

Available online 6 March 2007

Abstract

We discuss the foundations of factor or regression models in the light of the self-consistency condition that the market portfolio (and more generally the risk factors) is (are) constituted of the assets whose returns it is (they are) supposed to explain. As already reported in several articles, self-consistency implies correlations between the return disturbances. As a consequence, the alphas and betas of the factor model are unobservable. Self-consistency leads to renormalized betas with zero effective alphas, which are observable with standard OLS regressions. When the conditions derived from internal consistency are not met, the model is necessarily incomplete, which means that some sources of risk cannot be replicated (or hedged) by a portfolio of stocks traded on the market, even for infinite economies. Analytical derivations and numerical simulations show that, for arbitrary choices of the proxy which are different from the true market portfolio, a modified linear regression holds with a non-zero value α_i at the origin between an asset i's return and the proxy's return. Self-consistency also introduces "orthogonality" and "normality" conditions linking the betas, alphas (as well as the residuals) and the weights of the proxy portfolio. Two diagnostics based on these orthogonality and normality conditions are implemented on a basket of 323 assets which have been components of the S&P500 in the period from January 1990 to February 2005. These two diagnostics show interesting departures from dynamical self-consistency starting about 2 years before the end of the Internet bubble. Assuming that the CAPM holds with the self-consistency condition, the OLS method automatically obeys the resulting orthogonality and normality conditions and therefore provides a simple way to self-consistently assess the parameters of the model by using proxy portfolios made only of the assets which are used in the CAPM regressions. Finally, the factor decomposition with the self-consistency condition derives a risk-factor decomposition in the multi-factor case which is identical to the principal component analysis (PCA), thus providing a direct link between model-driven and data-driven constructions of risk factors. This correspondence shows that PCA will therefore suffer from the same limitations as the CAPM and its multi-factor generalization, namely lack of out-of-sample explanatory power and predictability. In the multi-period context, the self-consistency conditions force the betas to be time-dependent with specific constraints.

Keywords: Asset pricing; No arbitrage; Equilibrium; CAPM; APT; Market portfolio; Self-consistency; PCA

*Corresponding author also at the Institute of Geophysics and Planetary Physics, Department of Earth and Space Sciences, Los Angles, CA 90095, USA.
E-mail addresses: ymalevergne@ethz.ch (Y. Malevergne), dsornette@ethz.ch, dsornette@ethz.ch (D. Sornette).

0378-4371/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2007.02.076

1. Introduction

One of the most important achievements in financial economics is the capital asset pricing model (CAPM), which is probably still the most widely used approach to relative asset valuation. Its key idea is that the expected excess return of an asset is proportional to the expected covariance of the excess return of this asset with the excess return of the market portfolio. The proportionality coefficient measures the average relative risk aversion of investors. As a consequence, there is an irreducible risk component which cannot be diversified away, which cannot be eliminated through portfolio aggregation and thus has to be priced. The central testable implication of the CAPM is that assets must be priced so that the market portfolio is mean-variance efficient [1,2]. However, past and recent tests have rejected the CAPM as a valid model of financial valuation. In particular, the Fama/French analysis [3,4] shows basically no support for the CAPM's central result of a positive relation between expected return and global market risk (quantified by the beta parameter). In contrast, other variables, such as the market capitalization and the book-to-market ratio or the turnover and the past return, present some explanatory power.

More and more sophisticated extensions of the CAPM beyond the mean-variance approach have not improved the ability of the CAPM and its generalization to explain relative asset valuations. Let us mention the multi-moment CAPM, which has originally been proposed by Rubinstein [5] and Krauss and Litzenberger [6] to account for the departure of the returns distributions from Normality. The relevance of this class of models has been underlined by Lim [7] and Harvey and Siddique [8] who have tested the role of the asymmetry in the risk premium by accounting for the skewness of the distribution of returns and more recently by Fang and Lai [9] and Hwang and Satchell [10] who have introduced a four-moment CAPM to take into account the letpokurtic behavior of the assets return distributions. Many other extensions have been presented such as the VaR-CAPM [11], the Distributional-CAPM [12], and generalized CAPM models with consistent measures of risks and heterogeneous agents [13], in order to account more carefully for the risk perception of investors.

The arbitrage pricing theory (APT) provides an alternative to the CAPM. Like the CAPM, the APT assumes that only non-diversifiable risk is priced. But, unlike the CAPM which specifies returns as a linear function of only systematic risk, the APT is based on the well-known observations that multiple factors affect the observed time series of returns, such as industry factors, interest rates, exchange rates, real output, the money supply, aggregate consumption, investor confidence, oil prices, and many other variables [14–16]. While observed asset prices respond to a wide variety of factors, there is much weaker evidence that equities with larger sensitivity to some factors give higher returns, as the APT requires. This weakness in the APT has led to further generalizations of factor models, such as the empirical Fama/French three-factor model [17], which does not use an arbitrage condition anymore. Fama and French started with the observation that two classes of stocks show better returns that the average market: (1) stocks with small market capitalization (''small caps'') and (2) stocks with a high book-value-to-price ratio (often ''value'' stocks as opposed to ''growth'' stocks).

What then survive of the fundamental ideas underlying the CAPM? A key remark is that, given a set of assets, what is literally tested is the efficiency of a specific proxy for the market portfolio together with the CAPM. As recalled by [1], the CAPM requires using the market portfolio of all the invested wealth (which includes stocks, bonds, real-estate, commodities, etc.). More precisely, as first stressed by Roll [2], ''The theory is not testable unless the exact composition of the true market portfolio is known and used in the tests. This implies that the theory is not testable unless *all* individual assets are included in the sample.'' (italics in [2]). Unfortunately, the market proxies used in empirical work are almost always restricted to common stocks, and as pointed out by Roll, the composition of a proxy for the market portfolio can cause quite confusing inferences on the validity of the test and the mean-variance efficiency of the market portfolio. It is thus possible that the CAPM holds, the true market portfolio is efficient, and empirical contradictions of the CAPM are due to bad proxies for the market portfolio. Given a universe of N assets, it is always possible to construct a mean-variance portfolio (or any multi-moment generalization thereof), which will be such that the expected excess return of an asset is proportional to the expected covariance of the excess return of this asset with the excess return of the mean-variance portfolio. This results mechanically (or algebraically) from the construction of the mean-variance portfolio. While this property looks identical to the central test of the CAPM, in order for the CAPM to hold and for such a mean-variance portfolio to be the market portfolio, it should remain a mean-variance portfolio ex ante (out-of-sample). The failure of the CAPM together with such a construction for the proxy of the market portfolio is revealed by the notorious instability of mean-variance portfolios (see for instance [18]) with their weights needing to be continuously readjusted as a function of time. Empirically, the problem is that a mean-variance portfolio constructed over a given time interval will be no more in general a mean-variance portfolio (even allowing for a different average return) in the next period, and cannot thus qualify as the market portfolio.

In addition to this problem of the market portfolio proxy, the "disturbances" in factor models are correlated, as a consequence of the self-consistency condition that, in a complete market, the market portfolio and, more generally, the explanatory factors are made of (or can be replicated by) the assets they are intended to explain [48] (see also Sharpe's Nobel lecture [19]). This presence of correlations between return residuals may *a priori* pose problems in the pricing of portfolio risks: only when the return residuals can be averaged out by diversification can one conclude that the only non-diversifiable risk of a portfolio is born by the contribution of the market portfolio which is weighted by the beta of the portfolio under consideration. Previous authors have suggested that this is indeed what happens in economies in the limit of a large market $N \to \infty$, for which the correlations between residuals vanish asymptotically and the self-consistency condition seems irrelevant. For example, while Sharpe [19] concluded that, as a consequence of the self-consistency condition, at least two of the residuals, say ε_i and ε_j, must be negatively correlated, he suggested that this problem may disappear in economies with infinitely many securities. In fact, we show in Ref. [20] that this apparently quite reasonable line of reasoning does not tell the whole story: even for economies with infinitely many securities, when the companies exhibit a large distribution of sizes as they do in reality, the self-consistency condition leads to the important consequence that the risk born out by an investor holding a well-diversified portfolio does not reduce to the market risk in the limit of a very large portfolio, as usually believed. A significant proportion of "specific risk" may remain which cannot be diversified away by a simple aggregation of a very large number of assets. Moreover, this non-diversifiable risk can be accounted for in the APT by an additional factor associated with the self-consistency condition.

Here, our more modest goal is to present a review of the foundation of factor models using the self-consistent condition as a pivot to organize the presentation and form threads across different results scattered in the literature. Our goal will be reached if the reader starts to appreciate, as the authors did in the course of their digestion of the literature leading to some new results reported in [20], the many subtle issues interconnecting the concepts of equilibrium, no-arbitrage and risk pricing. In the physicist language, these concepts describe ultimately what can probably be seen as the attractive fixed point (equilibrium) of self-organizing systems with feedbacks. We believe that the study of the inner-consistency of these models can be useful to inspire the development of novel approaches addressing the above issues and others.

The organization of the paper is the following. In the next section, we consider an equilibrium model where the assets return dynamics can be explained by a single factor, the market. At equilibrium, this model is consistent with the CAPM but, due to the self-consistency condition that the market portfolio is constituted of the assets whose returns it is supposed to explain, the parameters of the original factor model remain unobservable. Only the CAPM betas are observable if the true market portfolio is known. Due the self-consistency condition, the residuals of the regression of the assets' returns with respect to the market portfolio can only be defined with a zero intercept. Then, the orthogonality condition obtained in Fama [48] concerning the disturbances of the factor models is derived both for a one-factor as well as for a multi-factor model. In Section 3, we discuss the calibration issues associated with the one factor model in relation with the impact of the non-observability of the actual market factor. We illustrate that, if a proxy is used (which is the real-life situation), then one can only measure a modified beta value which may differ from the true beta. In addition, a non-zero 'alpha' appears, which has, however, nothing to do with the unobservable alpha of the original factor model, but reflects the difference between the proxy and the market portfolio. Section 4 addresses the same question for multi-factor models. A multi-factor analysis with the self-consistency condition is shown to be equivalent to the principal component analysis (PCA) applied to baskets of assets. In the light of these results, Section 5 offers a discussion of the theoretical and practical limitations of the factor-models. It underlines the necessity for the introduction of non-constant β's and propose some restrictions on the possible dynamics for the β. All the technical derivations are gathered in the six appendices.

2. Self-consistency of factor models

2.1. One-factor model: dynamical consistency of the CAPM

2.1.1. Factor model from CAPM

The celebrated CAPM, derived by Sharpe [49], yields the famous relation known as the *market security line*:

$$\mathrm{E}[r_i] = r_0 + \beta_i \cdot \mathrm{E}[r_m - r_0], \tag{1}$$

where r_m, r_i and r_0 denote the market return, the return[1] on asset i and the risk free interest rate, respectively, while

$$\beta_i = \frac{\mathrm{Cov}(r_i, r_m)}{\mathrm{Var}\, r_m}. \tag{2}$$

As stressed by Sharpe [19], "the value β_i can be given an interpretation similar to that found in regression analysis utilizing historic data, although in the context of the CAPM it is to be interpreted strictly as an *ex ante* value based on probabilistic beliefs about future outcomes." If the investors' anticipations are self-fulfilling, the relationship between r_i and r_m can be modeled as

$$r_i = a_i + \beta_i \cdot r_m + \varepsilon_i, \tag{3}$$

with $a_i = (1 - \beta_i) r_0$, provided that the expectation of the disturbances $\mathrm{E}[\varepsilon_i]$ is assumed to be zero. These two conditions $a_i = (1 - \beta_i) r_0$ and $\mathrm{E}[\varepsilon_i] = 0$ ensure that the market portfolio is efficient in the mean-variance sense. Indeed, taking expectations (or sample means) of (3), one obtains an exact linear cross-sectional relation between mean returns and betas. There is a one-to-one correspondence between exact linearity and mean/variance efficiency of the market portfolio [21].

2.1.2. CAPM from a factor model

Let us now start from the opposite viewpoint to determine the conditions under which the CAPM relation holds for an economy obeying a linear factor model, where the excess returns of asset prices over the risk-free rate r_0 are determined according to the following equation[2]:

$$\vec{r}_t = \vec{\alpha} + \vec{\beta}^0 \cdot r_m(t) + \vec{\varepsilon}_t, \tag{4}$$

where $\vec{r}_t$ is the $N \times 1$ vector of asset excess returns at time t, $r_m(t)$ is the excess return on the market portfolio and $\vec{\varepsilon}_t$ is a vector of disturbances with zero average $\mathrm{E}[\vec{\varepsilon}_t] = \vec{0}$ and covariance matrix $\Omega_t = \mathrm{E}[\vec{\varepsilon}_t \cdot \vec{\varepsilon}_t']$. We assume that Ω_t is a deterministic function of t and that the $\vec{\varepsilon}_t$'s are independent through time. We do not make any other assumption concerning Ω_t, in particular, we do not assume that it is a diagonal matrix since the CAPM places no restriction on the correlation between the disturbance terms. The symbols $\vec{\alpha}$ and $\vec{\beta}^0$ represent constant $N \times 1$ vectors.

Let us assume that the model (4) is common knowledge, i.e., each economic agent knows that the asset returns follow Eq. (4), each agent knows that all other agents know that the assets returns follow equation (4), and so on... Let us assume that, by reallocating her wealth W_t among the N risky assets and the risk-free asset at each intermediate time period $t = 1, \ldots, T-1$, each agent aims at maximizing her expected terminal wealth W_T under the constraint that its variance $\mathrm{Var}\, W_T$ is not greater than a predetermined level $\sigma^2_{W_T}$. Mathematically, this dynamic optimization program reads

$$(\mathscr{P}): \begin{array}{ll} \max\limits_{\vec{w}} & \mathrm{E}[W_T] \\ \text{s.t} & \mathrm{Var}\, W_T \leqslant \sigma^2_{W_T}, \\ & W_{t+1} = W_t[1 + \vec{w}'\vec{r}_t + r_0], \quad t = 0, 1, \ldots, T-1. \end{array} \tag{5}$$

[1]Given the price $P_i(t)$ of security i at time t, its return is defined as $r_i(t) = P_i(t+1)/P_i(t) - 1$.

[2]In all what follows, we work with excess returns, i.e., returns decreased by the risk-free rate r_0 but use the same notation as for the returns to simplify the notations.

The term r_0 appears as a result of our convention to use returns defined as excess returns over the risk-free interest r_0. Many other approaches have been considered in the large body of literature devoted to the problem of optimal investment selection in a multi-period framework. In particular, the approaches based on the maximization of the expected utility of the terminal wealth or of the lifetime consumption seem to dominate, but they often rely on a specific choice of the utility function, such as the CARA, HARA or quadratic utility functions [22–24]. Since the choice of a particular utility function may appear as arbitrary, we have preferred to resort to the mean-variance criterion in so far as it constitutes a low order expansion approximation which holds irrespective of the specific form of the utility function.

The solution of problem ($\mathscr{P}$) can be found for instance in [25]: at each time period t, the optimal strategy amounts to invest a fraction of wealth in the risk free asset and the remaining in the risky portfolio

$$\vec{w}_t^* = \frac{\Sigma_t^{-1}\mathrm{E}[\vec{r}_t]}{\vec{1}'\Sigma_t^{-1}\mathrm{E}[\vec{r}_t]}, \tag{6}$$

where $\Sigma_t = \mathrm{Cov}\,\vec{r}_t$ denotes the covariance of the vector of excess returns of the asset prices over the risk-free rate, at time t. As we shall see in the sequel, Σ_t and $\mathrm{E}[\vec{r}_t]$ are known functions of t, which is a necessary assumption for the solution given by Li and Ng [25] to hold.

Since all agents invest only in two funds, namely the risk-free asset and the risky portfolio with weights $\vec{w}_t^*$, if we assume that an equilibrium is reached at each time t, then the composition $\vec{w}_t^*$ of the risky portfolio must represent that of the market portfolio at time t. In other words, in full generality, $\vec{w}_t^*$ given by (6) is nothing but the efficient tangency portfolio on the frontier composed of the existing risky assets. It becomes the market portfolio of all assets when the assets being considered here comprise indeed all assets, which is the case we first examine. Section 3 discusses what happens when this is not the case. For the sake of simplicity, we will denote by $\vec{w}_t$ the composition of the market portfolio thus dropping the sign $*$.

It is important to note that the result (6) holds irrespective of the time horizon T chosen by the investors because the composition $\vec{w}_t$ of the market portfolio is independent of T. Only the relative part of wealth invested in the risk-free asset and in the market portfolio depends on T, but this has no effect on the composition $\vec{w}_t$ of the market portfolio. As a consequence, the result still holds when investors have different time horizons, as in real markets.

Now, accounting for the fact that the market factor is itself built upon the universe of assets that it is supposed to explain (which we refer to as the "self-consistent condition"), the model must fulfill the internal consistency condition

$$r_m(t) = \vec{w}_t' \cdot \vec{r}_t. \tag{7}$$

Starting from this self-consistency condition together with the assumption that investors follow a dynamic mean-variance strategy and with the condition of market equilibrium, we show in Appendix A of Ref. [26] that the regression model (4) leads to the CAPM

$$\mathrm{E}[\vec{r}_t] = \vec{\beta}_t \mathrm{E}[r_m(t)], \tag{8}$$

with

$$\vec{\beta}_t = \frac{\mathrm{Cov}(\vec{r}_t, r_m(t))}{\mathrm{Var}\,r_m(t)} = \frac{(\vec{1}-\vec{\beta}^0)'\Omega_t^{-1}\vec{\alpha}}{\vec{\alpha}'\Omega_t^{-1}\vec{\alpha}}\vec{\alpha} + \vec{\beta}^0. \tag{9}$$

This shows that the regression model (4) is consistent with the relation of the CAPM provided that the internal consistency condition (7) holds together with the existence of an equilibrium.

The rather lengthy derivation in Appendix A of Ref. [26] is not needed in the standard approach in which the vector $\vec{\alpha}$ is identically zero and the market portfolio is mean-variance efficient as given by (6). Appendix A of Ref. [26] makes explicit that the parameters of the market model (4) are of no consequence for the CAPM. Appendix A of Ref. [26] derives the expression of the observable parameters of the CAPM (in particular the beta) from the parameters α's, β_0's and the matrix Ω of the covariance of the disturbances $\vec{\varepsilon}$ of the market model.[3]

[3] As we clarify further below, the disturbances $\vec{\varepsilon}$ of the market model are not the residuals of an OLS (ordinary least-square) regression.

Therefore, the general regression model (4) provides a reasonable statistical model to test the CAPM relation (8). But, two important point must be discussed. First, even if $\vec{\alpha}$ and $\vec{\beta}^0$ are assumed constant, the CAPM's β depends on time t as soon as Ω_t is not constant. Thus, the heteroscedasticity of the residuals is sufficient to make the β's time varying. Since, in the real market, the variance of assets returns is time varying (the so-called GARCH effect), one has to account for the dynamics of the β's. Second, the equilibrium imposes a dynamic constraint on the composition of the market portfolio. On the one hand, it is endogenously determined by the investors' anticipations according to formula (6). On the other hand, the market portfolio must be related to the market capitalization of each asset, which reflects the economic performance of the firms. Thus, the relation

$$w^i_{t+1} = w^i_t \cdot \frac{1 + r^i_t + r_0}{1 + r_m(t) + r_0} \tag{10}$$

must hold. The r_0 appears in the numerator and denominator because of our convention to denote by r^i_t and $r_m(t)$ the excess returns of asset and market prices over the risk-free interest r_0. For the time being, we assume that this relation (10) is compatible with the dynamics described by (4) and with the optimal portfolio allocation (6) and will discuss this point in more detail at the end of this article.

2.2. One-factor model: observable parameters, orthogonality and normalization conditions

For ease of the exposition, let us assume that Ω_t remains constant during the time interval under consideration. As a consequence, $\vec{\beta}$ can be *a priori* independent of t as shown by Eq. (9), allowing us to remove the subscript t in the sequel.

The previous sub-section has made clear that, according to (9), the coefficients $\vec{\beta}$ of the CAPM can be expressed in terms of the α's, β_0's and the matrix Ω of the covariance of the disturbances $\vec{\varepsilon}$ of the market model. Actually, one can go further and show that the self-consistency condition implies that only $\vec{\beta}_t$ is observable while the coefficients $\vec{\alpha}$ and $\vec{\beta}^0$ are unobservable. Indeed, expression (4) cannot be directly calibrated by the OLS estimator since the disturbances $\vec{\varepsilon}_t$ are correlated with the regressors while an OLS estimation automatically constructs residuals which are orthogonal to the factor decomposition. To see why the disturbances $\vec{\varepsilon}_t$ are correlated with the regressors $r_m(t)$, let us left-multiply expression (4) by $\vec{w}'_t$. Then, the self-consistency condition (7) implies that

$$r_m(t) = \frac{\vec{w}'_t \cdot (\vec{\alpha} + \vec{\varepsilon}_t)}{1 - \vec{w}'_t \cdot \vec{\beta}^0}, \tag{11}$$

unless $w_t\vec{\beta}^0 = 1$.

The fact that the regressors $r_m(t)$ are correlated with the residuals $\vec{\varepsilon}_t$ does not invalidate the OLS procedure. It just means that the OLS procedure will estimate residuals which are different from the model disturbances. The observed residuals are obtained by decomposing the disturbances $\vec{\varepsilon}_t$ on its component correlated with $r_m(t)$ plus a contribution uncorrelated with $r_m(t)$. We thus introduce two non-random vectors $\vec{\delta}$, $\vec{\gamma}$ and the random vector $\vec{u}_t$, *uncorrelated* with $r_m(t)$ with zero mean, such that

$$\vec{\varepsilon}_t = \vec{\delta} + \vec{\gamma} \cdot r_m(t) + \vec{u}_t. \tag{12}$$

Then, Appendix B of Ref. [26] shows that the one-factor model reduces to

$$\vec{r}_t = \vec{\beta} \cdot r_m(t) + \vec{u}_t, \tag{13}$$

with the ''normalization'' and ''orthogonality'' conditions

$$\vec{w}'_t\vec{\beta} = 1 \quad \text{and} \quad \vec{w}'_t\vec{u}_t = 0, \tag{14}$$

which derive from the self-consistency condition (7). The result (13) means that, *under the assumption that* $r_m(t)$ *is observable*, the OLS estimator of (4) provides an estimate of $\vec{\beta}$ *and not of* $\vec{\beta}^0$ *and* $\vec{\alpha}$ *which remain unobservable*. Taking the expectation of (13) recovers the CAPM prediction (8) as it should.

We should stress that the orthogonality condition $\vec{w}_t'\vec{u}_t = 0$ shows that at least two of the $u_{t,i}$ must be negatively correlated, which resemble Sharpe's [19] statement in his footnote 13. But, there is an important difference in that the regression (13) has zero intercept (its "alpha" is zero). The absence of intercept together with the mean-variance nature of the market portfolio automatically ensures the validity of the CAPM relation (8).

Using the jargon of physicists, we can rephrase these results as follows. The self-consistency condition together with the mean-variance efficient nature of the market portfolio implies that the market model (4) is "renormalized" into an observable model given by expression (13) with (14), that is, the "bare" parameters $\vec{\alpha}$ and $\vec{\beta}_0$ are renormalized into $\vec{0}$ and $\vec{\beta}$. A standard OLS regression (a measurement) gives access only to the renormalized values $\vec{0}$ and $\vec{\beta}$, in the same sense that physicists can only measure for instance the large scale renormalized mass and charge of an electron and not its bare values [27].

2.3. Multi-factor model

Let us generalize (4) and assume that the excess return vector $\vec{r}_t$ of n securities traded on the market (made of these N assets), over the risk free interest rate, can be explained by the q-factor model

$$\vec{r}_t = \sum_{i=1}^{q} \vec{\beta}_i u_i(t) + \vec{\varepsilon}_t \tag{15}$$

$$= B\vec{u}(t) + \vec{\varepsilon}_t, \tag{16}$$

where B is the $N \times q$ matrix which stacks the vectors $\vec{\beta}_i$, $\vec{u}_t$ is the vector whose ith component is the ith risk factor u_i and $\mathrm{E}[\vec{\varepsilon}(t)] = 0$.

With N assets and $N + q$ sources of randomness, the market is *a priori* incomplete. The market becomes complete if all risk factors can be replicated by an asset portfolio.

Consider the risk factor i, which can be replicated by the portfolio $\vec{w}_i$, that is, $u_i(t) = \vec{w}_i'\vec{r}_t$ in vector notations. The internal consistency of the model implies that

$$\vec{w}_i'\vec{r}_t = u_i(t) = \sum_{j=1}^{q} (\vec{w}_i'\vec{\beta}_j) u_j(t) + \vec{w}_i'\vec{\varepsilon}_t, \tag{17}$$

so that

$$\sum_{j \neq i} (\vec{w}_i'\vec{\beta}_j) u_j(t) + (\vec{w}_i'\vec{\beta}_i - 1) u_i(t) + \vec{w}_i'\vec{\varepsilon}_t = 0. \tag{18}$$

For a complete market such that all the risk factors u_i's can be replicated by asset portfolios $\vec{w}_i$'s, $i = 1, \ldots, q$, and denoting by W the matrix which stacks all the portfolio weight vectors $\vec{w}_i$'s, the self-consistency condition (18) generalizes to

$$(\mathrm{Id} - W'B)\vec{u}(t) = W'\vec{\varepsilon}_t. \tag{19}$$

Taking the expectation of both sides yields

$$(\mathrm{Id} - W'B)\mathrm{E}[\vec{u}(t)] = 0, \tag{20}$$

since we assume $\mathrm{E}[\vec{\varepsilon}(t)] = 0$. Two cases must be considered:

- First case: $\det(\mathrm{Id} - W'B) \neq 0$ and the unique solution is $\mathrm{E}[\vec{u}(t)] = 0$, so that $\mathrm{E}[\vec{r}_t] = 0$ by (16), which does not capture a real economy.
- Second case: $\det(\mathrm{Id} - W'B) = 0$, which means that the matrix $W'B$ has rank $q - p$, for some $0 < p \leqslant q$. Provided that the system admits a solution, this solution can be expressed as a linear combination of p independent vectors. As a consequence, the expected excess return on each individual asset $\mathrm{E}[r_i]$ can be

expressed as the linear combination of the expected value of only p risk factors. Therefore, only p factors really matter. This implies that, if we assume that assets excess returns really depend upon $p = q$ factors, the rank of the matrix $(\mathrm{Id} - W'B)$ should be $q - p = 0$ so that the expectation of the excess return on each individual asset $\mathrm{E}[r_i]$ can be expressed as the linear combination of the expected value of all the q risk factors. In such a case, we will say that the model is irreducible, an hypothesis that we will assume to hold in the sequel. The case $p < q$ can be treated analogously by expressing the excess return of each individual asset as a linear combination of the expected value of the p risk factors.

The condition that the rank of the matrix $(\mathrm{Id} - W'B)$ should be zero for the asset excess returns to depend on the q irreducible factors simply means that the normalization condition

$$W'B = \mathrm{Id} \tag{21}$$

must hold. This relation is satisfied by the market factor in the CAPM, and generalizes the normalization condition discussed in Section 2.1. In addition, Eq. (19) together with (21) enables us to conclude that

$$W'\vec{\varepsilon}_t = 0, \tag{22}$$

which means that the vector $\vec{\varepsilon}_t$ of disturbances has dimension $N - q$ at most, provided that W is full rank, i.e., provided that the q risk factors $u_i(t)$ can be replicated by q linearly independent portfolios $\vec{w}_i$. Condition (22) generalizes the orthogonality condition for the one-factor model derive in Section 2.2. The two conditions (21) and (22) generalize the orthogonality and normalization conditions (14) obtained for the one-factor CAPM.

Note that $\vec{u}$ and $\vec{\varepsilon}$ are uncorrelated under the condition that the q risk factors $u_i(t)$ can be replicated by q linearly independent portfolios.

To sum up, the possibility to replicate the risk factors by portfolios implies strong internal consistency conditions for factor models, namely Eqs. (21) and (22). Conversely, if these conditions are not met, the model is necessarily incomplete, which means that some sources of risk cannot be replicated (or hedged) by an asset portfolio. Therefore, risk factors, such as the GDP, the term spread, the dividend yield, the size and book-to-market factors [4,17] and so on, could bring in additional information with respect to the usual market factor. See Ref. [28] for empirical evidence.

3. Non-observability of the market portfolio (one-factor model)

3.1. What if the proxy is different from the true market portfolio?

In practice, the true market factor is unknown and one commonly uses a proxy. We show in Appendix C of Ref. [26] that model (13) leads to

$$\vec{r}_t = \underbrace{(\mathrm{E}[r_m]\vec{\beta} - \mathrm{E}[\tilde{r}_t]\vec{\tilde{\beta}})}_{\vec{\tilde{\alpha}}} + \vec{\tilde{\beta}} \cdot \tilde{r}_t + \vec{\eta}_t, \tag{23}$$

where $\tilde{r}_t$ is the proxy excess return, $\vec{\tilde{\beta}}$ is the vector of betas of the regression of asset excess returns on the proxy and $\vec{\eta}_t$ has zero mean $\mathrm{E}[\vec{\eta}_t] = 0$ and is uncorrelated with the proxy $\mathrm{Cov}(\vec{\eta}, \tilde{r}_t) = 0$. The explicit dependence of $\vec{\tilde{\beta}}$ as a function of the true $\vec{\beta}$, the weights $\tilde{w}'_t$ of the portfolio proxy, the variance $\mathrm{Var}\, r_m$ of the market portfolio excess returns and the covariance matrix $\tilde{\Omega}$ of the vector $\vec{u}_t$ of residuals of the model (13) is given by

$$\vec{r}_t = \frac{1}{\tilde{w}'\vec{\beta}} \vec{\beta} \cdot \mathrm{E}[\tilde{r}_t] + \underbrace{\frac{\tilde{\Omega}\tilde{w} + (\tilde{w}'\vec{\beta})\mathrm{Var}\, r_m \vec{\beta}}{(\tilde{w}'\tilde{\Omega}\tilde{w}) + (\tilde{w}'\vec{\beta})^2 \mathrm{Var}\, r_m}}_{\vec{\tilde{\beta}}} (\tilde{r}_t - \mathrm{E}[\tilde{r}_t]) + \vec{\eta}, \tag{24}$$

or, equivalently (see Appendix C of Ref. [26])

$$\vec{r}_t = \underbrace{\left(\frac{1}{\tilde{w}'\vec{\beta}}\vec{\beta} - \vec{\tilde{\beta}}\right)}_{-\vec{\gamma}} \cdot \mathrm{E}[\tilde{r}_t] + \vec{\tilde{\beta}} \cdot \tilde{r}_t + \vec{\eta}, \tag{25}$$

$$= (\mathrm{E}[r_m]\vec{\beta} - \mathrm{E}[\tilde{r}_t]\vec{\tilde{\beta}}) + \vec{\tilde{\beta}} \cdot \tilde{r}_t + \vec{\eta}. \tag{26}$$

The result (23) derives straightforwardly from the CAPM formulated explicitly with (13) and (14) by again using a self-consistent (or endogenous) condition that the proxy is itself a portfolio of the assets it is supposed to explain. As a consequence of the internal consistency requirement, one gets new orthogonality and normalization conditions. As previously, we have the normalization and orthogonality conditions

$$\tilde{w}_t'\vec{\tilde{\beta}} = 1 \quad \text{and} \quad \tilde{w}_t'\vec{\eta}_t = 0, \tag{27}$$

where $\tilde{w}_t$ represents the composition of the proxy at time t. In addition, we have the following orthogonality constraint:

$$\begin{aligned} \tilde{w}_t'\vec{\tilde{\alpha}} &= \tilde{w}_t'(\mathrm{E}[r_m]\vec{\beta} - \mathrm{E}[\tilde{r}_t]\vec{\tilde{\beta}}) \\ &= \underbrace{(\tilde{w}'\vec{\beta})}_{\beta \text{ of the proxy}} \quad \mathrm{E}[r_m] - \mathrm{E}[\tilde{r}_t] \\ &= 0, \end{aligned} \tag{28}$$

provided that the CAPM relation holds.

Using a proxy instead of the true market portfolio yields a non-vanishing intercept $\vec{\tilde{\alpha}} = \mathrm{E}[r_m]\vec{\beta} - \mathrm{E}[\tilde{r}_t]\vec{\tilde{\beta}}$ in the regression of the excess returns of each asset as a function of the excess returns of the portfolio proxy, which is *a priori* different from asset to asset. However, taking the expectation of (23), we obtain

$$\mathrm{E}[r_{i,t}] = \mathrm{E}[r_m]\beta_i = \left(\frac{\mathrm{E}[r_m]}{\mathrm{E}[\tilde{r}_t]}\frac{\beta_i}{\tilde{\beta}_i}\right)\mathrm{E}[\tilde{r}_t]\tilde{\beta}_i, \tag{29}$$

for each individual asset i. As in the standard CAPM prediction, we thus obtain that the expected excess return $\mathrm{E}[\vec{r}_{i,t}]$ of an asset i is proportional to its beta $\tilde{\beta}_i$ (obtained from the conditional regression (23)). But there is a major difference with the standard CAPM prediction, which is that the coefficient of proportionality is not simply the expectation $\mathrm{E}[\tilde{r}_t]$ of the proxy excess returns (as one could expect naively from translating the standard result to the proxy case). The difference involves the two correction factors $\mathrm{E}[r_m]/\mathrm{E}[\tilde{r}_t]$ and $\beta_i/\tilde{\beta}_i$, the second one being non-constant since it is a function of $\tilde{\beta}_i$ itself. Recall that $\mathrm{E}[r_m]$ and the β_i's are in principle unobservable. We can thus expect a deviation from the standard CAPM linear relationship due to an increased scatter induced by the scatter in the coefficient of proportionality between expected excess return and beta evaluated with a market proxy.

Although this result is generally true, there is an exception. If the proxy happens to be on the ex ante mean/variance efficient frontier, there will be an exact cross-sectional relation between expected returns and betas (calculated against the proxy) and there will be no scatter around the linear relation between mean returns and betas. Any market proxy will produce exact linearity, not just the tangency portfolio from the translated (by r_0) origin. Of course, the betas will be different for each such proxy but there will be no scatter. Generally, there is no need to assume the existence of a riskless rate. This is the heart of Black's [29] generalization of the CAPM. If there is no riskless rate, any ex ante mean-variance efficient portfolio, which can lie anywhere on the positive or negative part of the frontier, will produce exact cross-sectional mean return/beta linearity. The only exception is the global minimum variance portfolio, which is positively correlated with all assets. For all other market proxies, there is a "zero-beta" portfolio, a portfolio uncorrelated with the chosen proxy, which serves in place of the riskless rate.

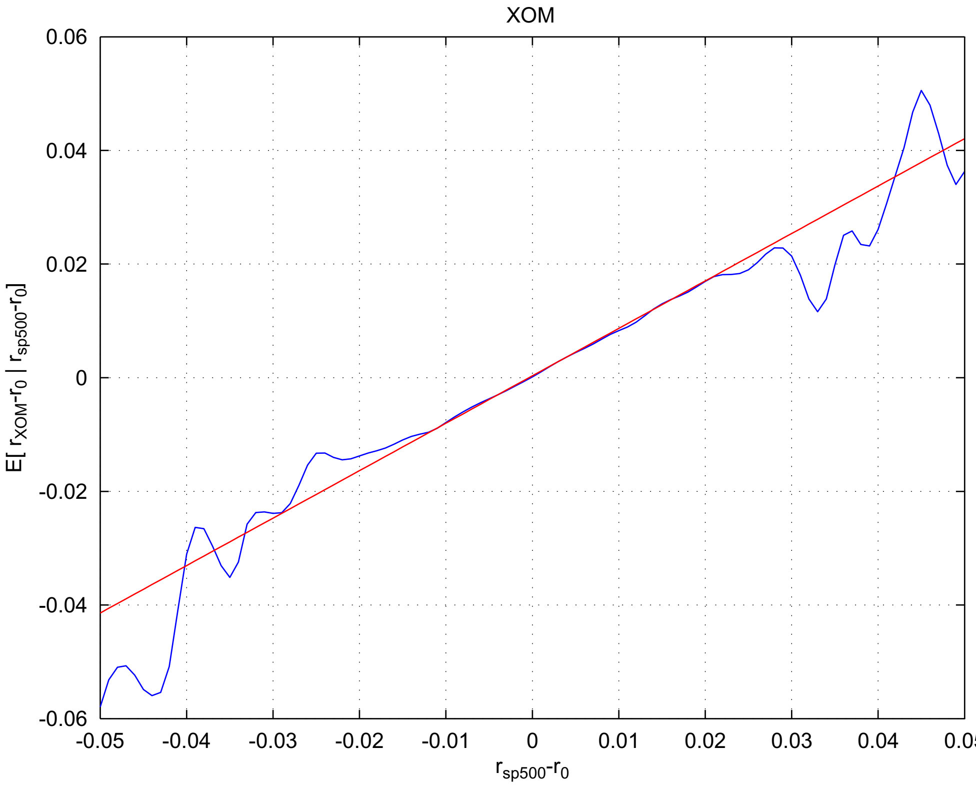

Fig. 1. Regression of the expected return above the risk-free interest rate for Exxon mobil daily returns with respect to the excess return of the S&P500 index over the period from July 1962 to December 2000. The risk free interest rate is obtained from the three month Treasury Bill.

3.2. Empirical illustration

As an illustration, let us first take the S&P500 index as a proxy for the USA market portfolio. Fig. 1 shows the average daily return of Exxon mobil (ticker XOM) daily returns conditioned on a fixed value of the S&P500 index daily returns $r_m(t)$ over the period from July 1962 to December 2000. In practice, we consider a given value r_m (within a small interval) of the S&P500. We then search for all days for which the return of the S&P500 was equal to this value r_m (within a small interval). We then take the average of the daily return of Exxon mobil realized in all these days. We then iterate by scanning all possible values of r_m and use a kernel estimation to get a smoother and more robust estimation. Note that this procedure is non-parametric and provides an interesting determination of the market model. Indeed, suppose that the return r_i of an asset i is given by

$$r_i(t) = F_i[r_m(t)] + e(t), \tag{30}$$

where $F_i[x]$ is an *a priori* arbitrary (possibly non-linear function) and $e(t)$ are the zero-mean residuals. Then, the above non-parametric procedure (whose result is shown in Figs. 1 and 2) amounts to calculate $\mathrm{E}[r_i|r_m = x]$ as a function of x:

$$\mathrm{E}[r_i|r_m = x] = F_i[x]. \tag{31}$$

Fig. 1 plots the function $F_i[x]$ determined non-parametrically from the data. It seems that a linear dependence provides a reasonable approximation of the data presented in Fig. 1. The straight line is the line of equation $y = \alpha_{\mathrm{XOM}} + \beta_{\mathrm{XOM}} \cdot r_{SP500}(t)$, where β_{XOM} is obtained from the regression

$$r_{\mathrm{XOM}}(t) = \alpha_{\mathrm{XOM}} + \beta_{\mathrm{XOM}} \cdot r_{SP500}(t) + \varepsilon_{\mathrm{XOM}}(t) \tag{32}$$

of the returns.

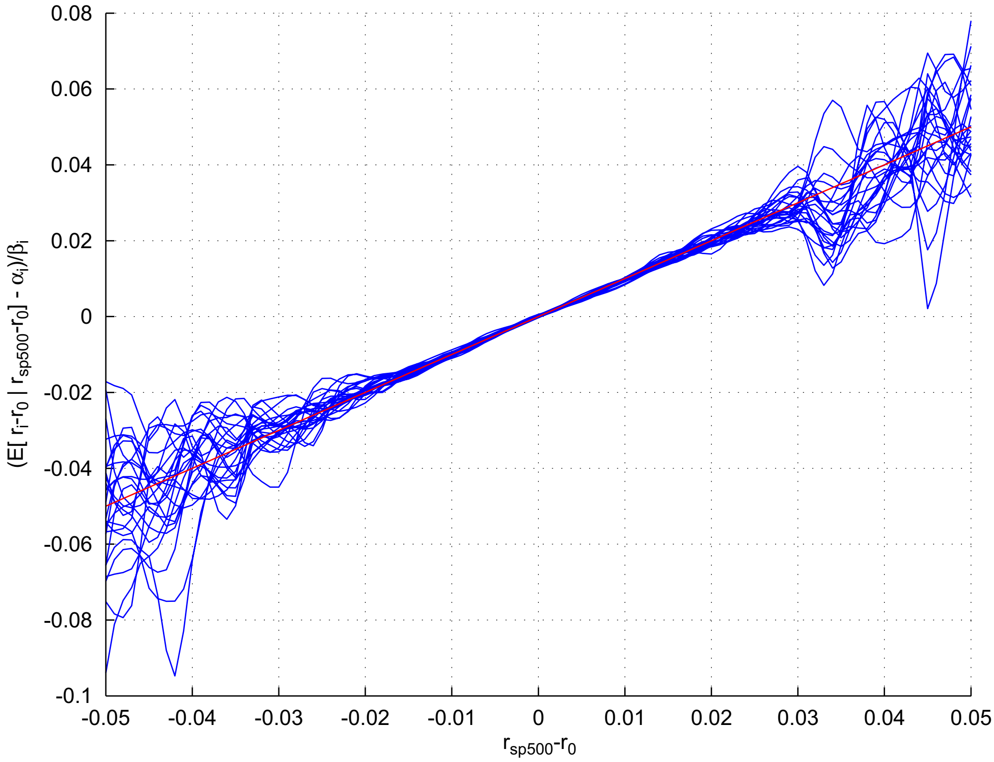

Fig. 2. Each curve is similar to that shown in Fig. 1 and represents the normalized expected return above the risk-free interest rate defined by (33) for a given stock i over the period from July 1962 to December 2000 as a function of the excess return $r_{SP500} - r_0$ above the risk-free interest rate r_0 of the S&P500 index taken as a proxy of the market portfolio. Since the α_i's and β_i's are different from asset to asset, the normalization (33) ensures by construction that a good linear regression for each asset should be qualified by having all curves collapse on the diagonal, with unit slope and crossing of the origin, as observed up to statistical fluctuations. The 25 curves corresponds to the following stocks: Abbott Laboratories, American Home Products Corp., Boeing Co., Bristol-Myers Squibb Co., Chevron Corp., Du Pont (E.I.) de Nemours & Co., Disney (Walt) Co., General Electric Co., General Motors Corp., Hewlett-Packard Co., International Business Machines Co., Coca-Cola Co., Minnesota Mining & MFG Co., Philip Morris Cos Inc., Merck & Co Inc., Pepsico Inc., Pfizer Inc., Procter & Gamble Co., Pharmacia Corp., Schering-Plough Corp., Texaco Inc., Texas Instruments Inc., United Technologies Corp., Walgreen Co. and Exxon Mobil Co. The risk free interest rate is obtained from the three month Treasury Bill.

This plot presented in Fig. 1 is typical of the relationship between conditional expected returns as a function of the return of the S&P500 index, obtained for all stocks in the S&P500, as shown from the superposed data in Fig. 2. Fig. 2 is the same as Fig. 1, but for 25 different assets. In order to represent the corresponding functions $F_i(x)$ for each asset on a same figure without loosing visibility, we have just translated and scaled each curve, i.e., we plot

$$\frac{\mathrm{E}[r_i - r_0 | r_{SP500} - r_0] - \alpha_i}{\beta_i} = (F_i[x] - \alpha_i)/\beta_i, \tag{33}$$

as a function of $x = r_{SP500} - r_0$, where the α_i's and β_i's are obtained by linear regressions similar to (32), one fit being performed for each non-parametrically determined F_i. The risk-free interest rate r_0 is basically negligible at the daily scale. $\mathrm{E}[r_i - r_0 | r_{SP500} - r_0]$ is the expected return of stock i above the risk-free interest rate, conditional on the value of $r_{SP500} - r_0$. The straight line in Fig. 2 has slope 1 and goes through the origin, thus confirming the remarkable quality of the relationship between the conditional expected asset returns and the S&P500 index daily returns, in agreement with (23). In other words, Fig. 2 seems to confirm that the F_i's appear to be quite closely approximated by an affine function: $F_i[x] = \alpha_i + \beta_i x$.

We have performed similar regressions as a function of the S&P500 returns for the monthly returns of the 323 stocks which remained into the composition of the S&P500 over the period between January 1990 and February 2005. But, in order to test the self-consistency condition and its consequences derived above, one

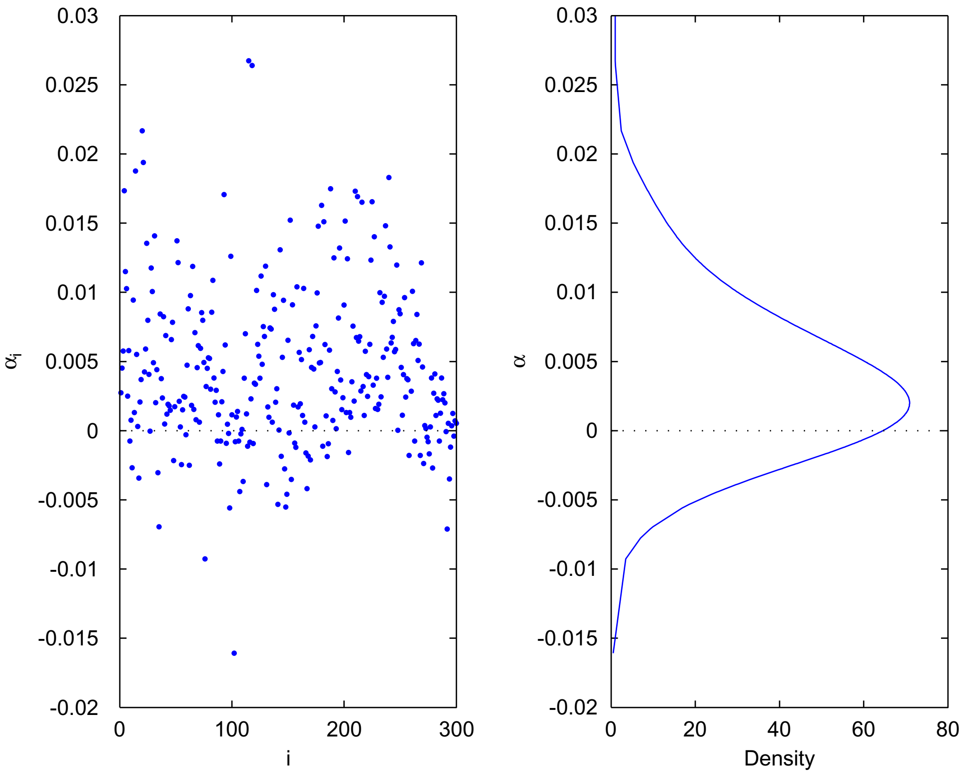

Fig. 3. Left panel: population of the intercepts of the regression of the expected monthly excess returns of 323 stocks entering into the composition of the S&P500 between January 1990 and February 2005 versus the monthly excess returns of the effective S&P323 index that we have constructed as a portfolio of these 323 stocks with weights proportional to their capitalizations. The risk free interest rate is obtained from the three month Treasury Bill. The abscissa is an arbitrary indexing of the 323 assets. The estimated probability density function of the population of alphas is shown on the right panel and illustrates the existence of a systematic bias for the alphas.

could argue that it should be better to construct a market portfolio based solely on these 323 stocks. We have thus constructed an effective S&P323 index, constituted as a portfolio of these 323 stocks with weights proportional to their capitalizations. The regressions of the expected monthly returns of each of these 323 stocks conditioned on the S&P323 index monthly returns as a function of the S&P323 index monthly returns are similar to those obtained on the S&P500 and resemble the regressions shown in Figs. 1 and 2 albeit with more noise (not shown). Fig. 3 shows the population of the intercepts (the alpha's) of these regression. The abscissa is an arbitrary indexing of the 323 assets. The estimated probability density function of the population of alphas is shown on the right panel and illustrates the existence of a systematic bias for the alphas, as expected from the previous Section 3.1. Note that the presence of a (positive) bias simply amounts to say that the constructed index is not located on the sample efficient frontier.

Fig. 4 plots the expected returns $E[r_i - r_0]$ of the monthly excess returns of the 323 assets used in Fig. 3 as a function of their β_i obtained by regressions with respect to the excess return of the effective S&P323 index. Under the CAPM hypothesis, one should obtain a straight line with slope $E[r_{SP323} - r_0]$ (0.62% per month) and zero additive coefficient at the origin. The straight line is the regression $y = 0.18\% + 0.89\% \cdot x$. A standard statistical test shows that the value 0.18% of the intercept at the origin is marginally not significantly different from zero at the 5% level. Together with the reasonable agreement between the slope of the regression and the excess expected returns of the S&P323 index, this would give a positive score for the CAPM. This is perhaps surprising considering the biases distribution of alphas shown in Fig. 3. This suggests that this standard expected return/beta tests exemplified in Fig. 4 has not large power.

As a complement, one can use the self-consistency conditions $\tilde{w}_t' \vec{\beta} = 1$ (expression 27) and $\tilde{w}_t' \vec{\alpha} = 0$ (expression 28) to perform empirical tests. As explained in Section 2.1, the dynamical consistency of the

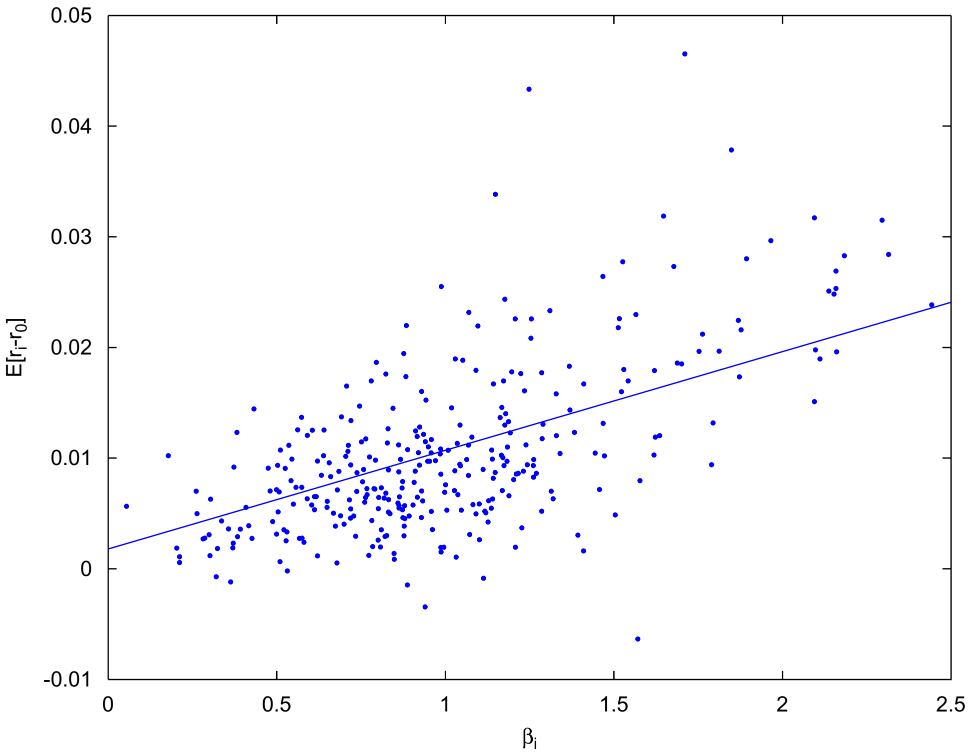

Fig. 4. Expectation $E[r_i - r_0]$ of the monthly excess returns of the 323 assets used in Fig. 3 as a function of their β_i obtained by regressions with respect to the excess return to the effective S&P323 index. The risk free interest rate is obtained from the three month Treasury Bill. Under the CAPM hypothesis, one should obtain a straight line with slope $E[r_{SP323} - r_0]$ (0.62% per month) and zero additive coefficient at the origin. The straight line is the regression $y = 0.18\% + 0.89\% \cdot x$.

CAPM imposes that these two relationships should hold at each time step for the proxy of the market portfolio. We have thus calculated $\tilde{w}_t'\vec{\beta}$ and $\tilde{w}_t'\vec{\alpha}$, where $\tilde{w}_t$ is the vector of weights of the 323 stocks in our effective S&P323 index which evolves at each time step according to the capitation of each stock while $\vec{\beta}$and $\vec{\alpha}$ are the two vectors of betas and alphas obtained from the regressions used in Figs. 3 and 4. Fig. 5 shows the time evolution of $\tilde{w}_t'\vec{\beta}$ and $\tilde{w}_t'\vec{\alpha}$ over the period from January 1990 to February 2005 which includes 182 monthly values. The deviations, respectively, from 1 and 0 are significant, as shown by a standard Fisher test. The close connection between the time varying average alpha and beta shown in Fig. 5 results from their common dynamics through the evolution of the weights $\vec{w}$.

The variable $\tilde{w}_t'\vec{\beta}$ can be interpreted as the average beta of the stocks in the self-consistent market proxy. A value different from 1 suggests that the market is out of equilibrium. In particular, if $\tilde{w}_t'\vec{\beta} > 1$, this can be interpreted as an "over-heating" of the market with the existence of positive feedback. Interestingly, this occurs just about two years before the peak of the Internet bubble in April 2000. It then took about two years after the peak to recover an equilibrium. Since early 2003, the market seems to have remained approximately at equilibrium according to this metric.

3.3. Tests on a synthetically generated market

In order to investigate the sensitivity of these tests, and in particular the impact of using a proxy for the market portfolio, we have constructed a toy (synthetic) market in which 1000 assets are traded and such that their returns at time t obey equation (13) with the constraints (14). The weight of each asset in the market portfolio is drawn from a power law with tail index equal to one, in accordance with empirical observations on

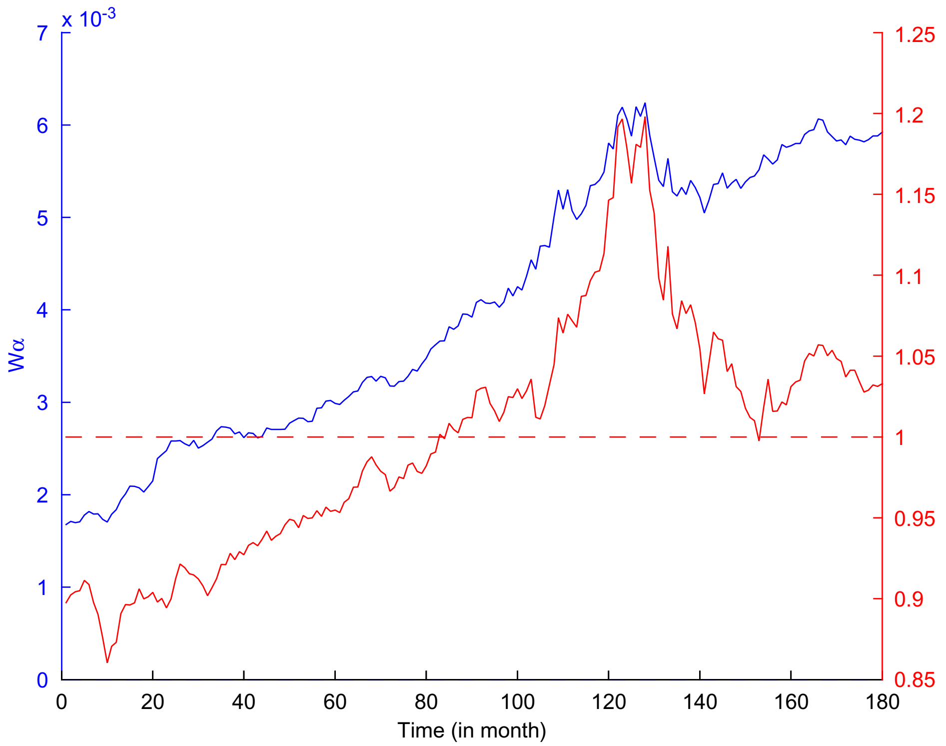

Fig. 5. Time evolution of $\tilde{w}_t'\vec{\beta}$ (red lower curve and right vertical scale) and $\tilde{w}_t'\vec{\alpha}$ (blue upper curve and left vertical scale) over the period from January 1990 to February 2005 which includes 182 monthly values. $\tilde{w}_t$ is the vector of weights of the 323 stocks in our effective S&P323 index which evolves at each time step according to the capitation of each stock. $\vec{\beta}$ and $\vec{\alpha}$ are the two vectors of betas and alphas obtained from the regressions used in Figs. 3 and 4. According to the self-consistency conditions (27) and (28), the dynamical consistency of the CAPM should lead to $\tilde{w}_t'\vec{\beta} = 1$ and $\tilde{w}_t'\vec{\alpha} = 0$ at all time periods.

the distribution of firm sizes [30], and then renormalized so that the weights sum up to one. For the purpose of illustration and easiness in testing, we impose that the composition of the market remain constant, i.e., the economy is stationary. The interest in this condition is that we can then study the pure impact of not observing the true market but only the proxy constructed on a subset of the whole universe of assets. The daily return on the synthetic market factor follows a Gaussian law with mean and standard deviation equal to the mean and the standard deviation of the daily return on the S&P500 over the time period from July 1962 to December 2000, namely 0.037% and 0.90%, respectively. The β's are also randomly drawn from a uniform law with mean equals to one and are such that they satisfy the normalization condition (14). It can be seen in Fig. 6 that the β's range between 0.35 and 1.15, which is reasonable if we refer to the values usually reported in the literature. Finally, the residuals $\vec{\varepsilon}_t$ are drawn from a degenerate multivariate Gaussian distribution (i.e., the rank of its covariance matrix is $N-1=999$), so that they fulfill the orthogonality condition (14). The variances and covariances of these residuals have been fixed in such a way that they are of the same order of magnitude as the variances and covariances of the residuals estimated by linear regression of our basket of 25 assets on the S&P500. Thus, the values given by our toy market are expected to be consistent with the values observed on the actual market if the description by a one factor model has some merit.

Using the OLS estimator, we have first performed a regression with respect to the true market portfolio, whose composition is assumed to remain constant as we said. Then, we have constructed an arbitrary portfolio and have considered it to be the proxy of the market portfolio. We have then performed the linear regression of the assets returns on the proxy returns. Fig. 6 compares the estimated betas obtained from the regression of the asset returns on the returns of the market portfolio with those obtained from the regression on the returns of the proxy, as a function of the true betas. The regression on the market factor gives a line

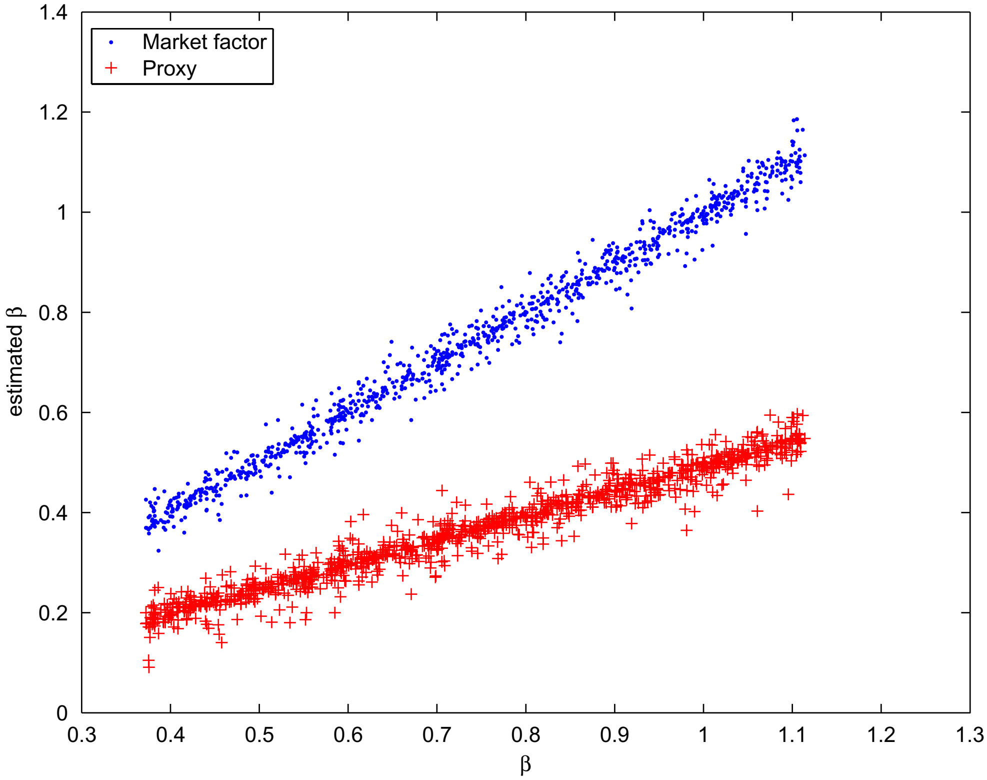

Fig. 6. Synthetic tests on an artificial market of 1000 synthetic assets with properties adjusted to mimic those of the real US market. The plot shows the estimated betas obtained from the regression of the asset returns on the returns of the market portfolio (blue dots) and on the returns of the proxy (red crosses), as a function of the true betas. The upper straight line corresponds to the ideal case where the estimated betas equal the true betas. The lower straight line is the predicted dependence (24) of the betas estimated with the proxy as a function of the true beta.

with unit slope and zero intercept, as expected from the construction of the synthetic market. The regression on the proxy returns gives also a straight line, as predicted from the linear relation between $\vec{\beta}$ and $\vec{\tilde{\beta}}$ given by (24). Fig. 6 provides a verification of the properties put by construction in our synthetic market. Obviously, no one would be able to perform this verification on real data since the market portfolio and thus the true betas are unknowable.

Fig. 7 shows the population of the intercepts of the regression of expected stock returns versus the market return or versus the proxy return in our synthetic market. These intercepts are presented as a function of the (arbitrary) indices of the 1000 assets. For the regression on the market factor, one can observe as expected a scatter around zero. For the regression on the market proxy, the intercepts are, on average, all significantly different from zero. As expected, the orthogonality and normalization conditions $\tilde{w}'\vec{\alpha} = 0$ and $\tilde{w}'\vec{\tilde{\beta}} = 1$ are satisfied, providing a verification of the validity of the numerical implementation of the model for these synthetically generated data. Thus, Fig. 7 confirms that a universe of assets which by construction obeys the CAPM exhibits non-zero alpha intercepts (which take apparently random values) when using an arbitrary proxy. This result can be compared with the empirical analog shown in Fig. 3.

Fig. 8 shows the individual expected returns $E[r_i]$ for each of the 1000 assets (i) as a function of the true β_i's, (ii) as a function of the β_i's obtained by regression on the true market and (iii) by regression on the proxy. As expected, the dependence of the expected returns on the true betas and on the betas obtained from the true market portfolio follows the CAPM prediction, but with rather significant fluctuations. The scatter of the dependence of the expected returns on the betas determined from the proxy is larger but one can still observe a well-defined linear dependence with a zero intercept, and a slope different from the expected return $E[\tilde{r}_t]$ of the

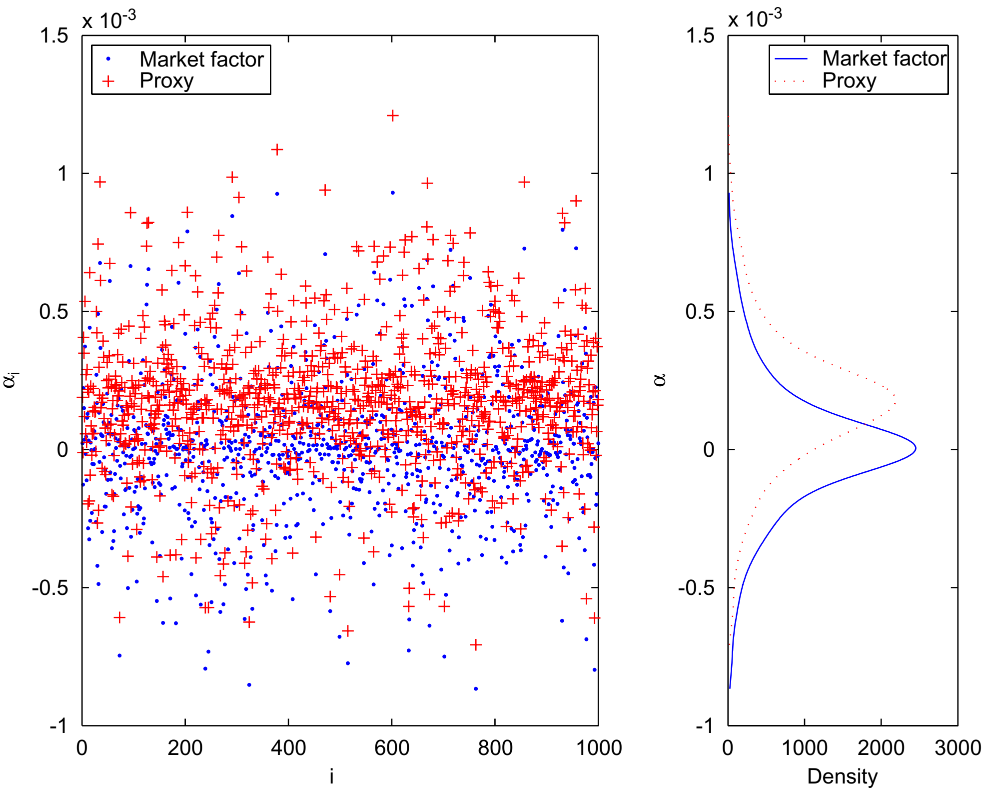

Fig. 7. Synthetic tests on an artificial market of 1000 synthetic assets with properties adjusted to mimic those of the real US market. Left panel: Population of the intercepts of the regression of expected stock returns versus the market return (blue dots) or versus the proxy return (red crosses) in our synthetic market. The abscissa is an arbitrary indexing of the 1000 assets of our artificial market. The estimated probability density functions of the two population of alphas are shown on the right panel and illustrate the existence of a systematic bias for the proxy's alphas.

portfolio proxy, as predicted in expression (29). This seems to justify why the bias in the distribution of alphas does not seem to affect the existence of the standard expected return/beta test shown in Fig. 3.

3.4. *On the orthogonality and normality conditions*

To summarize, the condition of self-consistency leads to the orthogonality and normality conditions (14) for the mono-factor model and to (21,22) for the multifactor model when the market portfolio is known. The orthogonality and normality conditions still hold when only a market proxy is available and they take the form (27) together with the additional orthogonality constraint (28). This suggests to use the orthogonality and normality conditions as new tests of the CAPM in the real-life situation where the market portfolio is not known and a somewhat arbitrary proxy is used. The motivation of these tests stems from the fact that they are not affected by the problem of using a proxy which is different from the real market factor, in contrast with the problem on the standard test of the CAPM made explicit in Fig. 8. Concretely, this suggests to complement the standard expected excess return versus beta, by tests checking the validity of the orthogonality and normality conditions when using for the proxy, *not* the S&P500, but *any* portfolio constructed on the assets used in the test. A test of the CAPM would then consist in testing the normalization and orthogonality conditions (27)–(28), which should hold for any such proxy portfolio.

It turns out however that the OLS estimated intercepts $\hat{\vec{\alpha}}$, the estimated β's $\hat{\vec{\beta}}$ and the estimated residuals $\hat{\vec{\eta}}$ of a basket of assets necessarily satisfy the constraints (27)–(28) when the proxy used as the regressor is a portfolio build on these same assets. Let us denote by Y the matrix which stacks the returns of the basket of the N assets under consideration, by X the matrix of the regressors, by B the matrix of the regression

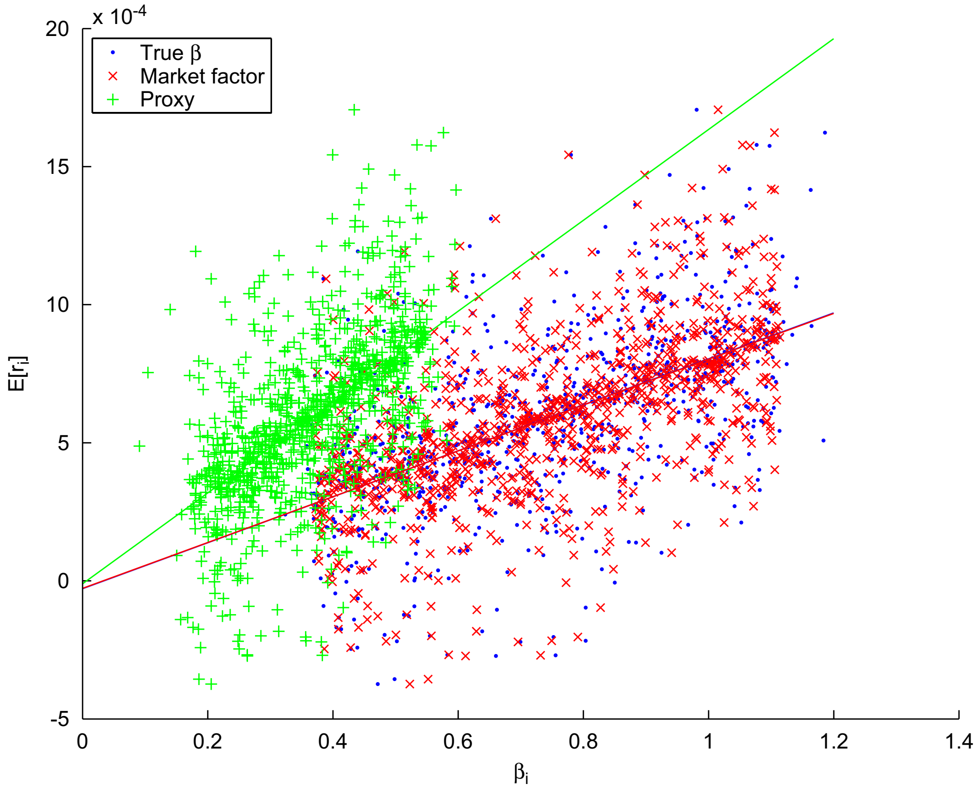

Fig. 8. Synthetic tests on an artificial market of 1000 synthetic assets with properties adjusted to mimic those of the real US market. Individual expected returns $E[r_i]$ for each of the 1000 assets (i) as a function of the true β_i's (blue dots), (ii) as a function of the β_i's obtained by regression on the true market (red crosses $\times$) and by regression on the proxy (green $+$). The straight lines are the linear regressions.

coefficients and by U the matrix which stacks the vectors of the residuals:

$$Y = \begin{pmatrix} \vec{r}'_1 \\ \vdots \\ \vec{r}'_T \end{pmatrix}, \quad X = \begin{pmatrix} 1 & r_m(1) \\ \vdots & \vdots \\ 1 & r_m(T) \end{pmatrix}, \quad B = \begin{pmatrix} \alpha_1 & \cdots & \alpha_N \\ \beta 1 & \cdots & \beta_N \end{pmatrix}, \quad U = \begin{pmatrix} \vec{\eta}'_1 \\ \vdots \\ \vec{\eta}'_T \end{pmatrix}, \tag{34}$$

so that, if $\vec{r}_m$ denotes the vector of the returns on any portfolio W made of our N assets only, we have

$$\begin{pmatrix} r_m(1) \\ \vdots \\ r_m(T) \end{pmatrix} = YW. \tag{35}$$

With these notations, the linear regression equation reads $Y = XB + U$. The OLS estimators of B and of U are then, respectively,

$$\hat{B} = X^{\mathrm{t}} X^{-1} X^{\mathrm{t}} Y, \tag{36}$$

and

$$\hat{U} = Y - X\hat{B} = [\mathrm{Id} - X(X^{\mathrm{t}}X)^{-1}X^{\mathrm{t}}]Y. \tag{37}$$

It is then easy to show that

$$\hat{B}W = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \hat{U}W = \vec{0}, \tag{38}$$

which are nothing but the constraints (27)–(28) in matrix form. Their derivation involves the same kind of algebraic manipulations as those employed in Appendix E of Ref. [26] discussed in the next section and are thus not repeated here. Therefore, given any portfolio made of the subset of assets under consideration only, the OLS estimator automatically provides estimates which fulfill the self-consistency constraints. This prevents us from using these constraints as a way to test the CAPM. However, this derivation shows that, assuming that the CAPM holds, the OLS method provides a simple way to self-consistently assess the parameters of the model by using proxy portfolios made only of the assets which are used in the CAPM regressions.

4. Multi-factor models

4.1. Orthogonality and normality conditions

Extending Section 3.1, we now investigate the implications of using portfolio proxies for the explanatory factors in the multi-factor model analyzed in Section 2.3.

Let us first assume that the individual asset returns can be explained by *exactly* q factors. Then, q factor proxies are built by defining q portfolios of the traded assets. Let us denote by $\tilde{W}$ the matrix whose columns represent the q portfolios and by $\vec{v}_t$ the vector of the q proxies. Appendix D of Ref. [26] shows that, similarly to the result (23) obtained for the one-factor model, a non-zero intercept $\vec{\alpha}$ appears in the regression of the vector of asset returns with respect to the q proxies in the vector $\vec{v}_t$. In addition, the normalization condition

$$\tilde{W}'\tilde{B} = \mathrm{Id} \tag{39}$$

and the two orthogonality conditions

$$\tilde{W}'\vec{\alpha} = \vec{0} \quad \text{and} \quad \tilde{W}'\vec{v}_t = \vec{0} \tag{40}$$

hold, where v_t is the vector of the residuals of the multivariate regression on the vector of the q proxies v_t.

A priori, we do not know how many factors are needed but there are standard tests in factor analysis that provide some estimates of the number of factors [31,32]. It is possible to encounter a situation where the number r of portfolio proxies is different from the true number q of factors. The case $r<q$ corresponds to market incompleteness. Let us discuss the situation where $r>q$. In this case, Eqs. (39) and (40) still hold, as shown in Appendix E of Ref. [26], but a difficulty arises from the fact that the matrix $\tilde{W}'B$ is not a $q \times q$ matrix anymore, it is a $r \times q$ matrix, where $r>q$ is the number of chosen factor proxies. As a consequence, $(\tilde{W}'B)^{-1}$ does not exist and has to be replaced by its (left) pseudo-inverse. As previously, a non-zero intercept $\vec{\alpha}$ also appears in the regression of the vector of asset returns with respect to the q proxies. The orthogonality and normalization conditions still hold, as shown in Appendix E of Ref. [26].

4.2. Self-consistent calibration of the multi-factor model and principal component analysis (PCA)

Let us assume the existence of Q factors which can be replicated by Q portfolios W_i (the market is complete). Let W be the matrix which stacks all these portfolios: $W = (W_1, W_2, \ldots, W_Q)$. We again denote $\vec{r}_t$ as the vector of excess returns of the N assets over the risk free rate,[4] $\vec{u}_t = W'\vec{r}_t$ is the set of factors and B is the matrix of betas. This defines the model (16):

$$\vec{r}_t = B\vec{u}_t + \vec{\varepsilon}_t \tag{41}$$

$$= BW'\vec{r}_t + \varepsilon_t, \tag{42}$$

[4] If for instance the APT is true (i.e., there are no arbitrages available), then one does not need to subtract means for the intercept in (42) to be zero.

where the intercept is set to zero, which is always possible provided that we subtract the mean value of $\vec{r}_t$. Appendix F of Ref. [26] shows how to estimate the betas B and the Q replicating portfolios $W = (W_1, W_2, \ldots, W_Q)$ by using the properties

$$W'\vec{1}_N = \vec{1}_Q, \tag{43}$$

$$W'B = \mathrm{Id}_Q, \tag{44}$$

$$W'\vec{\varepsilon}_t = 0. \tag{45}$$

The first property (43) just expresses the normalization of the portfolio weights. The two other properties are the normalization and orthogonality conditions derived from the self-consistency condition that the factors can be replicated by portfolios constituted of the assets that they are supposed to explain (see (14) for the one-factor case and (21),(22) for the multi-factor case).

Appendix F of Ref. [26] first derives the relation

$$W = B[B'B]^{-1}, \tag{46}$$

between the matrix W of weights and the matrix B of betas, showing the dependence between W and B resulting from the self-consistency conditions. Finally, B and W can be constructed as

$$B = P'U\nabla V', \tag{47}$$

$$W = P'U\nabla^{-1}V'. \tag{48}$$

The matrix P is specified by the decomposition $RR' = P'DP$ where $R = (\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_T)$ is an $N \times T$ matrix and D is the diagonal matrix with elements equal to the eigenvalues of RR' and P is the matrix of the (orthogonal) eigenvectors of RR'. The matrix U is also fixed to

$$U = \begin{pmatrix} \mathrm{Id}_Q \\ 0 \end{pmatrix}, \tag{49}$$

i.e., it has its first Q upper diagonal elements equal to 1 and all its other elements equal to zero. The matrix V is not uniquely fixed, reflecting in this way the rotational degeneracy of the Q factors. Indeed, the matrix V can be any $Q \times Q$ orthogonal matrix whose lines add up to a non-vanishing constant.

Expression (42) with (47), (48) offers a practical decomposition of the market risks, using a multi-factor model generalizing the CAPM. It is useful to compare it with other available methods. It is customary in the financial literature to distinguish between model-driven and data-driven constructions of risk factors [33]. The CAPM is a good example of a model-driven method which imposes strict relationship between asset prices. On the other hand, the principal components analysis (PCA) method is the archetype of data-driven methods, which enjoys widespread use among statistical practitioners [34,35]. PCA is frequently employed to reduce the data dimensionality to a tractable value without needing strong hypotheses about the nature of the data generating process. Now, the reader familiar with PCA will notice that expression (42) with (47,48) provides a decomposition of risk components which *is nothing but* the decomposition obtained by using PCA! In other words, this section together with Appendix F of Ref. [26] has shown that a multi-factor analysis implemented *with the self-consistency condition* is equivalent to the empirical methodology of analyzing baskets of assets using PCA.

In general, there is no necessary connection between data-driven and model-driven constructions of risk factors. But, as soon as one uses a factor model, if the factors can be indeed expressed in terms of the assets themselves they are supposed to explain (as in the Fama/French 3-factor model) which is nothing but the self-consistency condition, then it follows automatically and necessarily that there is a connection between the factor model and the PCA: in fact, the factor analysis and the PCA are one and the same. This shows again the strong constraint that the self-consistency condition provides. This provides a direct link between model-driven and data-driven constructions of risk factors: one of the best representative of model-driven risk factor decomposition methods (the multi-factor model with self-consistency) is one and the same as one of the best examples of data-driven risk factor decomposition methods (the PCA). This correspondence implies that PCA will therefore suffer from the same limitations as the CAPM and its multi-factor generalization, namely lack of out-of-sample explanatory power and predictability. The exact correspondence between self-consistent multi-factor models and PCA justifies claims on the empirical and practitioner literature[5] that PCA may be an implementation of the arbitrage pricing theory (APT) [14–16]. Our result also suggests that using PCA to pre-filter the data before a factor decomposition is misconceived since both PCA and factor decomposition are one and the same thing. It might, however, be useful in non-linear factor decomposition, as suggested from previous non-linear dynamic studies [36–38].

PCA is theoretically better in one sense: it works with the raw covariance matrix of returns and hence should uncover any factors present in that matrix. The same cannot be said about approaches in terms of a fixed pre-determined number of factors. It is quite possible that the later approaches will fail to uncover important factors. However, PCA has a disadvantage because it is difficult to estimate when allowing for time variation in the true covariance matrix. It is in that sense that the factor models are more tractable.

5. Discussion and conclusion

We have structured the presentation of factor models in the light of the self-consistency condition. Starting from arbitrary factor models, internal consistency requirements have been shown to impose strong constraints on the coefficients of the factor models. These requirements merely express the fact that the factors employed to explain the changes in assets prices are themselves combinations of these securities. These conditions read

$$W_t' B_t = \mathrm{Id} \quad \text{and} \quad W_t' \vec{\varepsilon}_t = 0. \tag{50}$$

In addition, when proxies of the market factors are used instead of the factors themselves, a non-vanishing intercept $\vec{\alpha}$ appears which satisfies the third constraint

$$W' \vec{\alpha} = 0. \tag{51}$$

These constraints are appealing and it would have been natural to use them to test the adequacy of the factor-models. However, they are automatically fulfilled by the regression (i) on a proxy which is a portfolio whose composition is constant through time and is restricted to the subset of assets under consideration and (ii) on the factors derived from the PCA, when one uses this statistical method to select the relevant explaining factors. Thus, on the one end, these constraints do not allow to test the CAPM (or the multi-factor models), which remains untestable unless the entire market is considered, as first stressed by Roll [41]; nevertheless, on the other hand, the OLS estimator and the PCA provides a consistent method to assess the value of the different parameters of the problem.

Now, to escape from this self-referential approach which consists in regressing the assets returns on the returns on a portfolio made of the assets under consideration with constant proportion, one has to use a proxy with non-constant composition, such as the Standard & Poor's 500 index. In such a case, the normalization and orthogonality conditions (50)–(51) must hold at each time t. Thus, for a number of periods t larger than the number N of assets constituting the proxy, the number of constraints is larger than the number of parameters α_i's and β_i's to estimate. This implies that $\vec{\beta}$ and $\vec{\alpha}$ cannot be constant, unless the time varying vectors of market weights $\vec{w}_t$ "live" in a subspace of $\mathbb{R}^N$ which is orthogonal to $\vec{\alpha}$ and such that $\vec{w}_t' \cdot \vec{\beta} = 1$ (given by (14) for the mono-factor model, by (21,22) for the multifactor model when the market portfolio is known and by (27,28) when only a market proxy is available).

This condition raises questions on the dynamic consistency of the CAPM. As stressed, and then immediately swept under the carpet, at the end of Section 2.1, the equilibrium imposes a dynamic constraint on the composition of the market portfolio: on the one hand, it is endogenously determined by the investors' anticipations according to formula (6) while, on the other hand, the market portfolio must be related to the market capitalization of each asset, which reflects the economic performance of the industry. Thus, the relation (10) must hold. It can be rewritten as

$$w_{t+1}^i = w_t^i \cdot \frac{1 + r_0 + \beta_i r_m(t) + u_t^i}{1 + r_0 + r_m(t)}. \tag{52}$$

[5] See for instance http://www.perfectdownloads.com/business-finance/investment-tools/pickstock.htm and http://www.apt.com/en/aboutus/theaptapproach.html

This relation would be compatible with the normalization condition at times t and $t+1$ if and only if $\sum_{i=1}^{N}\beta_i w_t^i = \sum_{i=1}^{N}\beta_i w_{t+1}^i = 1$ which would imply that

$$r_m(t) = \frac{\sum_{i=1}^{N}\beta_i w_t^i u_t^i}{1 - \sum_{i=1}^{N} w_t^i \beta_i^2}. \tag{53}$$

But now, what could justify such a relation between the market return and the residuals. They have been assumed independent (or at least uncorrelated) up to now. Recall that our basic assumption was that $r_m(t)$ is exogenously fixed by the economic environment.

In this respect, it seems imperative to give up the assumption of a constant $\vec{\beta}$. But, as a consequence, it becomes necessary to specify a dynamics for $\vec{\beta}_t$. Several works have started addressing this question [39–45] and have proved the merit of this approach. With regard to this question, both Eq. (9) and Figs. 1 and 2 suggest the existence of a well-defined average β. Besides, considering that the volatility of the assets returns is mean-reverting, which is a well-known stylized fact [46,47], Eq. (9) shows that such an assumption should also hold for the dynamic of β_t.[6]

Finally, the normalization condition shows that $\vec{\beta}_t$ can be written as the sum of two terms

$$\vec{\beta}_t = \frac{\vec{w}_t}{||\vec{w}_t||^2} + \vec{\beta}_t^{\dagger}, \tag{54}$$

where $\vec{w}_t' \cdot \vec{\beta}_t^{\dagger} = 0$. The first term, $\vec{w}_t/||\vec{w}_t||^2$, is directly related to the Herfindahl index, i.e., the concentration, of the market portfolio. So, everything else taken equal, the risk premium increases when the level of diversification of the market decreases. As a first approximation, $\vec{\beta}_t^{\dagger}$ could be taken constant, so that the dynamics of $\vec{\beta}_t$ could be easily related to the dynamics of the market portfolio, which is a predictable quantity ($\vec{w}_t$ is known at time $t-1$, by use of (52)).

As mentioned briefly in the introduction, there is another interesting consequence of the self-consistency condition when an addition ingredient holds, namely when the distribution of the capitalization of firms is sufficiently heavy-tailed. In such case which seems to be relevant to real economies, assuming that a general complete equilibrium with no-arbitrage holds, then there may exist a new source of significant systematic risk, which has been totally neglected up to now but must be priced by the market [20]. This result is based on the self-consistency condition discussed at length in this paper, which leads mechanically to correlations between return residuals which are equivalent to the existence of a new "self-consistency" factor. Then, when the distribution of the capitalization of firms is sufficiently heavy-tailed, it is possible to show, using methods associated with the generalized central limit theorem, that the "self-consistency" factor does not disappear even for infinite economies and may produce significant non-diversified non-priced risks for arbitrary well-diversified portfolios. For economies in which the return residuals are function of the capitalization of firms, the new self-consistency factor provides a rationalization of the SMB (Small Minus Big) factor introduced by Fama and French. Accounting for the fact that high book-to-market stocks have significantly lower beta's with respect to the market portfolio compared with low book-to-market stocks, the book-to-market factor also seems to emerge naturally from our formalism.

Acknowledgment

The authors acknowledge helpful discussions and exchanges with R. Roll. All remaining errors are ours.

[6] To get this result, let us start from expression (9) for the vector $\vec{\beta}$. Let us assume that the matrix Ω has a dynamics of its own which is mean-reverting, $\Omega(t) = \Omega_0 + f(t)O$, where we assume that the time dependence is in the scalar factor $f(t)$, while O is a constant matrix. Let us assume that $f(t)$ is small, so that $f(t)O$ constitutes a perturbation to Ω_0. Expression (9) can be expanded to first order in powers of $f(t)$ to obtain $\beta(t) = C(1 + f(t)\bar{O}) + \beta_0$, where C and $\bar{O}$ are constant matrices which can be expressed in terms of $O, \Omega_0, \vec{\alpha}$ and $\vec{\beta}_0$. This shows that, if $f(t)$ is mean-reverting, then $\beta(t)$ is also mean-reverting.

Appendix A. Supplementary data

Supplementary data associated with this article can be found in the on-line version at doi:10.1016/j.physa.2007.02.076.

References

[1] E.F. Fama, K.R. French, The CAPM: theory and evidence, J. Econ. Perspect. 18 (2004) 25–46.
[2] R. Roll, A critique of the asset pricing theory's tests, part I: On past and potential testability of the theory, J. Finan. Econ. 4 (1977) 129–176.
[3] E.F. Fama, K.R. French, The cross-section of expected stock returns, J. Finance 47 (1992) 427–465.
[4] E.F. Fama, K.R. French, Common risk factors in the returns on stocks and bonds, J. Finan. Econ. 33 (1993) 3–56.
[5] M. Rubinstein, The fundamental theorem of parameter-preference security valuation, J. Finan. Quant. Anal. 8 (1973) 61–69.
[6] A. Krauss, R. Litzenberger, Skewness preference and the valuation of risk assets, J. Finance 31 (1976) 1085–1099.
[7] K.G. Lim, A new test for the three-moment capital asset pricing model, J. Finan. Quant. Anal. 24 (1989) 205–216.
[8] C.R. Harvey, A. Siddique, Conditional skewness in asset pricing tests, J. Finance 55 (2000) 1263–1295.
[9] H.B. Fang, T. Lai, Co-kurtosis and capital asset pricing, Finan. Rev. 32 (1997) 293–307.
[10] S. Hwang, S. Satchell, Modelling emerging market risk premia using higher moments, Int. J. Finance Econ. 4 (1999) 271–296.
[11] G.J. Alexander, A.M. Baptista, Economic implications of using a mean-VaR model for portfolio selection: a comparison with mean-variance analysis, J. Econ. Dynam. Control 26 (2002) 1159–1193.
[12] V. Polimenis, On the concavity of jump equity premia, Finan. Lett. 3 (1) (2005) (paper 18, February).
[13] Y. Malevergne, D. Sornette, Multi-moments method for portfolio management: generalized capital asset pricing model in homogeneous and heterogeneous markets, in: B. Maillet, E. Jurczenko (Eds.), Multi-moment Asset Allocation and Pricing Models, Wiley, New York, 2006, pp. 165–193.
[14] S.A. Ross, The arbitrage theory of capital asset pricing, J. Econ. Theory (December), (1976) 341-60.
[15] R. Roll, S.A. Ross, The arbitrage pricing theory approach to strategic portfolio planning, Finan. Analy. J. (May/June) (1984) 14–26.
[16] R. Roll, What every CFO should know about scientific progress in financial economics: what is known and what remains to be resolved, Finan. Manage. 23 (2) (1994) 69–75.
[17] E.F. Fama, K.R. French, Size and book-to-market factors in earnings and returns, J. Finance 50 (1995) 131–155.
[18] R.A. Michaud, A practical framework for portfolio choice, J. Invest. Manage. 1 (2) (2003) 1–16.
[19] W.F. Sharpe, Capital asset prices with and without negative holdings, Nobel Lecture, 1990.
[20] Y. Malevergne, D. Sornette, A Two-Factor Asset Pricing Model and the Fat Tail Distribution of Firm Sizes, Working Paper, ETH Zurich, 2006.
[21] Z. Bodie, A. Kane, A.J. Marcus, Investments, sixth ed., McGraw-Hill, Irwin, New York, 2004.
[22] P.A. Samuelson, Lifetime portfolio selection by dynamic stochastic programming, Rev. Econ. Statist. 50 (1969) 239–246.
[23] N.H. Hakansson, On optimal myopic portfolio policies, with and without serial correlation of yields, J. Bus. 44 (1971) 324–334.
[24] S.R. Pliska, Introduction to Mathematical Finance, Blackwell, Malden, MA, 1997.
[25] D. Li, W.L. Ng, Optimal dynamic portfolio selection: multiperiod mean-variance formulation, Math. Finance 10 (2000) 387–406.
[26] Y. Malevergne, D. Sornette, Self-consistent asset pricing models; Supplementary data, available in the on-line version of this paper at doi:10.1016/j.physa.2007.02.076.
[27] E.M. Lifshitz, L.P. Pitaevskii, V.B. Berestetskii, Quantum Electrodynamics, second ed., vol. 4, Butterworth-Heinemann, London, 1982.
[28] R. Petkova, Do the Fama–French factors proxy for innovations in predictive variables? J. Finance 61 (2006) 581–612.
[29] F. Black, Capital market equilibrium with restricted borrowing, J. Bus. 45 (1972) 444–455.
[30] R.L. Axtell, Zipf distribution of U.S. firm sizes, Science 293 (2001) 1818–1820.
[31] G. Connor, R. Korajzcyk, A test for the number of factors in an approximate factor model, J. Finance 48 (4) (1993) 1263–1291.
[32] J. Bai, S. Ng, Determining the number of factors in approximate factor models, Econometrica 70 (1) (2002) 191–221.
[33] M. Loretan, (1997) Generating market risk scenarios using principal components analysis: methodological and practical considerations, in: The Measurement of Aggregate Market Risk, CGFS Publications No. 7, pp. 23–60. Bank for International Settlements, November 1997. Available at ⟨http://www.bis.org/publ/ecsc07.htm⟩.
[34] G.H. Dunteman, Principal Components Analysis, Sage, California, 1989.
[35] I.T. Jolliffe, Principal Component Analysis, second ed., Springer Series in Statistics, New York, 2002.
[36] D.S. Broomhead, G. King, Extracting qualitative dynamics from experimental data, Physica D 20 (1986) 217–236.
[37] R. Vautard, P. Yiou, M. Ghil, Singular-spectrum analysis: a toolkit for short, noisy chaotic signals, Physica D 58 (1992) 95–126.
[38] K.S. Chan, H. Tong, Chaos: a statistic perspective, Spring, New York, 2001.
[39] M.E. Blume, On the assessment of risk, J. Finance 26 (1971) 1–10.
[40] M.E. Blume, Betas and their regression tendencies, J. Finance 30 (1975) 785–795.
[41] J. Ohlson, B. Rosenberg, Systematic risk of the CRSP equal-weighted common stock index: a history estimated by stochastic parameter regression, J. Bus. 55 (1982) 121–145.

[42] C.F. Lee, C.R. Chen, Beta stability and tendency: an application of the variable mean response regression model, J. Econ. Bus. 34 (1982) 201–206.
[43] T. Bos, P. Newbold, An empirical investigation of the possibility of stochastic systematic risk in the market model, J. Bus. 57 (1984) 35–41.
[44] R. Simmonds, L. La Motte, A. McWhorter, Testing for nonstationarity of market risk: an exact test and power considerations, J. Finan. Quant. Anal. 21 (1986) 209–220.
[45] D.W. Collins, J. Ledolter, J. Rayburn, Some further evidence on the stochastic properties of systematic risk, J. Bus. 60 (1987) 425–448.
[46] S. Satchell, J. Knight, Forecasting Volatility in the Financial Markets, second ed., Quantitative Finance, Butterworth-Heinemann, London, 2002.
[47] S. Figlewski, Forecasting Volatility, Blackwell, Oxford, 2004.
[48] E.F. Fama, A note on the market model and the two-parameter model, J. Finance 28 (1973) 1181–1185.
[49] W. Sharpe, Capital asset prices: a theory of market equilibrium under condition of risk, J. Finance 19 (1964) 425–442.