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Analysis of price diffusion in financial markets using PUCK model

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Abstract

Based on the new type of random walk process called the potentials of unbalanced complex kinetics (PUCK) model, we theoretically show that the price diffusion in large scales is amplified $2(2+b)^{-1}$ times, where b is the coefficient of quadratic term of the potential. In short time scales the price diffusion depends on the size M of the super moving average. Both numerical simulations and real data analysis of Yen–Dollar rates are consistent with theoretical analysis.

Keywords: Market prices; Potential force; Hurst exponent

1. Introduction

Crashes and uncontrollable hikes can often occur in financial markets. Such changes of the prices confuse the market and damage the economy because they start abruptly in many cases. Therefore, techniques to measure the probabilistic risk of sudden change in the prices have been studied using tick-by-tick data [1]. Recently, it was empirically found that change of prices can be approximated by the Fokker–Planck equation and the new type of random walk in a potential field [2–5]. The potential field is approximated by a quadratic function with its center given by the moving average of past market prices. This random walk model is called the potentials of unbalanced complex kinetics (PUCK) model in which the potential slowly changes in the market [3,4]. In this paper, we focus on the diffusion properties of this random walk process and calculate the diffusion coefficient which is helpful for estimating the market risk.

We first review an empirical derivation of the PUCK model. We next show that the statistically steady condition of price fluctuations depends on the potential field, and clarify relationships between the price diffusion and the potential field. We finally demonstrate that the price diffusion in short time scales depends on the size of moving average, however, large scale diffusion properties are independent of the moving average. In the paper, we used all the Bid record (about 20 million ticks) of the exchange rates for Yen/Dollar that were traded by the term from 1989 to 2002 to find the firm statistical laws.

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2. Empirical derivation of PUCK model

Prices in financial markets always have violent fluctuation in a short time scale. We first eliminate the uncorrelated noise $\eta(t)$ from the price $P(t)$ in order to reduce the statistical error. We next investigate the dynamics of the price.

We can perform this noise elimination process by introducing an optimum moving average $\overline{P(t)}$:

$$P(t) = \overline{P(t)} + \eta(t), \tag{1}$$

$$\overline{P(t)} = \sum_{k=1}^{K} w_k \cdot P(t-k), \tag{2}$$

where $P(t)$ is a price, $\eta(t)$ is an uncorrelated noise and w_k gives the weight factors where the time is measured by ticks. The weight factors are calculated by using Yule–Walker equation [6–8]. In a case of Yen–Dollar rate, the weight factor w_k follows an exponential function whose characteristic decay time is about 30 s [6].

We investigate a behavior of the optimum moving average $\overline{P(t)}$ obtained by eliminating the uncorrelated noise $\eta(t)$ from the price $P(t)$. We introduce a super moving average $\overline{P_M(t)}$ defined by

$$\overline{P_M(t)} = \frac{1}{M}\sum_{\tau=0}^{M-1} \overline{P(t-\tau)}. \tag{3}$$

In financial markets, it is found that the following relationship between $\overline{P(t)}$ and $\overline{P_M(t)}$ holds for a certain range of M [2]:

$$\overline{P(t+1)} - \overline{P(t)} = -\frac{1}{2}\cdot\frac{b(t)}{M-1}\cdot\frac{\mathrm{d}}{\mathrm{d}\overline{P}}\left(\overline{P(t)} - \overline{P_M(t)}\right)^2 + f(t), \tag{4}$$

where the mean of noise $f(t)$ is zero. Eq. (4) means that the price change can be approximated by a random walk in a quadratic potential field whose center is given by the moving average of past prices. It is known that the potential coefficient $b(t)$ has a long autocorrelation [3,4].

3. Statistically steady condition of price fluctuations in the PUCK model

We focus on Eq. (4) with the case of a constant b because the coefficient $b(t)$ is known to change slowly in financial markets. Eq. (4) is transformed as follows:

$$\overline{P(t+1)} - \overline{P(t)} = -\frac{b}{2}\left(\frac{2}{M(M-1)}\sum_{k=1}^{M-1}(M-k)\left(\overline{P(t-k+1)} - \overline{P(t-k)}\right)\right) + f(t). \tag{5}$$

This is a type of AR process for price difference when b is a constant. We can estimate the conditions of b to make the AR process being statistically steady. Eq. (5) is transformed by the following determinant:

$$\begin{aligned} X_t &= AX_{t-1} + F_t \\ &= (A)^t X_0 + (F_t + AF_{t-1} + \cdots + (A)^{t-1}F_1), \end{aligned} \tag{6}$$

where

$$X_t = \begin{pmatrix} \overline{P(t+1)} - \overline{P(t)} \\ \overline{P(t)} - \overline{P(t-1)} \\ \overline{P(t-1)} - \overline{P(t-2)} \\ \vdots \\ \overline{P(t-M+3)} - \overline{P(t-M+2)} \end{pmatrix}, \quad A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{M-2} & \alpha_{M-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

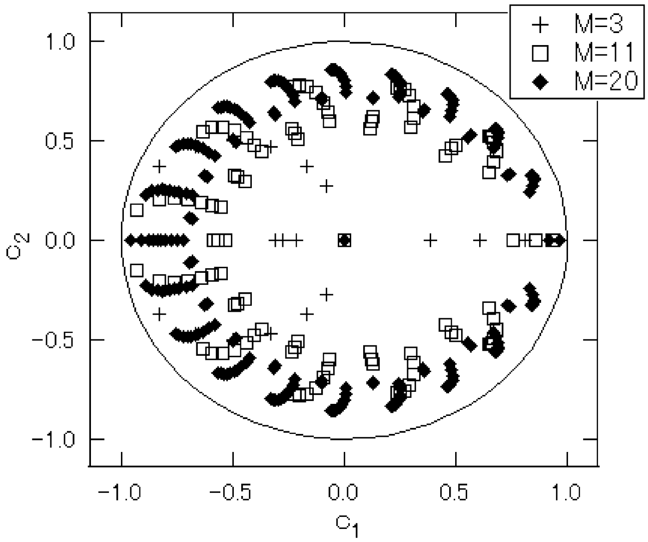

Fig. 1. The eigenvalues λ of matrix A. The c_1 and c_2 are real and imaginary parts of λ. The pluses are λ for $M=3$ when $-2<b<6$. The squares show λ for $M=11$ when $-2<b<22$. The diamonds express λ for $M=20$ when $-2<b<38$. The circle indicates $|\lambda|=1$.

$$F_t = \begin{pmatrix} f(t) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \alpha_k = -\frac{b(M-k)}{M(M-1)}. \tag{7}$$

If $\lim_{t\to\infty}(A)^t = 0$, the time series of $\overline{P(t+1)} - \overline{P(t)}$ is in a weakly steady state because X_t becomes independent of the initial value of X_0. This condition is fulfilled when the absolute values of all eigenvalues of A are less than 1. For example, when $M=2$ the time series of $\overline{P(t+1)} - \overline{P(t)}$ is statistically steady if the potential coefficient satisfies $|b|<2$. When $M=3$ the eigenvalues of A are given by

$$\lambda_1 = \frac{\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_2}}{2}, \quad \lambda_2 = \frac{\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}}{2}. \tag{8}$$

Solving the steady state condition we find that the potential coefficient should be in the range of $-2<b<6$ for $M=3$. Numerically calculating the eigenvalues for $2\leqslant M\leqslant 20$ as shown in Fig. 1 we find that the time series of $\overline{P(t+1)} - \overline{P(t)}$ is statistically stationary if the potential coefficient is in the following range:

$$\begin{cases} -2<b<2(M-1) & \text{when } M \text{ is even number,} \\ -2<b<2M & \text{when } M \text{ is odd number.} \end{cases} \tag{9}$$

Outside the condition of Eq. (9), the range of price fluctuations increases indefinitely depending on the time t.

4. Diffusion of prices in market potential field

As the potential coefficient $b(t)$ has a long autocorrelation, we can calculate the future price diffusion using Eq. (4). This prediction is crucial in order to evaluate the risks of market. We clarify statistical laws of price diffusion described by Eq. (4) using both simulations and theoretical analysis.

By simulating Eq. (4) for the case of $b(t)$ is a constant, we investigate the standard deviation on a time scale T defined by

$$\sigma_b(T) = \sqrt{\langle(\overline{P(t+T)} - \overline{P(t)})^2\rangle}. \tag{10}$$

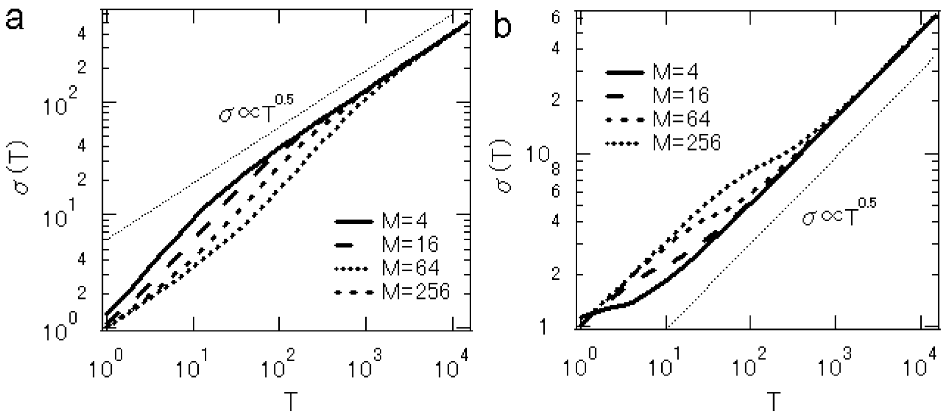

Fig. 2. Standard deviation $\sigma(T)$ of price changes in the time scale T. The heavy lines show price diffusions of numerical simulation for $M=4$ (top), 16 (second), 64 (third), 256 (bottom). Here, $b=-1.5$ in (a) and $b=2$ in (b). The standard deviation of $f(t)$ is 1. The guideline indicates $\sigma(T)\propto T^{0.5}$.

In Fig. 2 we plot $\sigma_b(T)$ for $M=4$, 16, 64, 256 when $b=-1.5$ and 2. Here, $f(t)$ is the Gaussian random number whose standard deviation is 1. The time scale where the Hurst exponent converges 0.5 depends on M. For example, the time scale is about 10^3 when $M=256$, while the time scale is around $T=20$ when $M=4$.

We can estimate the standard deviation $\sigma_b(T)$ in the long time scale limit. The variance of change of optimum moving average price $\overline{P(t)}$ is given by

$$\langle(\overline{P(t+T)}-\overline{P(t)})^2\rangle = -\left(\frac{b}{2}\right)^2\left(\frac{2}{M(M-1)}\right)^2\left\langle\left(\sum_{k=1}^{M-1}(M-k)(\overline{P(t-k+T)}-\overline{P(t-k)})\right)^2\right\rangle$$
$$-b\frac{2}{M(M-1)}\left\langle\sum_{k=1}^{M-1}((M-k)(\overline{P(t+T)}-\overline{P(t)})(\overline{P(t-k+T)}-\overline{P(t-k)}))\right\rangle$$
$$+\left\langle\left(\sum_{\tau=0}^{T-1}f(t+\tau)\right)^2\right\rangle. \tag{11}$$

By introducing a rough approximation

$$\langle(\overline{P(t+T)}-\overline{P(t)})\cdot(\overline{P(t+T-k)}-\overline{P(t-k)})\rangle \approx \langle(\overline{P(t+T)}-\overline{P(t)})^2\rangle, \tag{12}$$

we have the following simple formulation after some calculation:

$$\sigma_b(T)=\left(\frac{2}{2+b}\right)\sigma_{b=0}(T), \tag{13}$$

where $T\gg M$, $\sigma_b(T)=\sqrt{\langle(\overline{P(t+T)}-\overline{P(t)})^2\rangle}$ and $\sigma_{b=0}(T)$ is the standard deviation when $b=0$. In this long time scales, the standard deviation of the price is amplified $2(2+b)^{-1}$ times by the potential field. In Fig. 3 we show relationships between the potential coefficient b and the ratio of standard deviation $\sigma_b(T)/\sigma_{b=0}(T)$ when $T=10^5$ by simulating Eq. (4) for $M=2$, 4, 8, 16, 32. We can confirm that the price diffusion of numerical simulations follows Eq. (13) independent of M in the long time scale. From Eq. (13) we can theoretically find that in the long time scales the price diffusion is independent of M, independent of b, $\sigma_b(T)\propto T^{0.5}$.

5. Diffusion of Yen–Dollar rates

From the data set of real market prices, we can estimate the value of b, and presume the best value of M by comparing the price diffusion of numerical simulations to real price diffusion. Fig. 4 shows the diffusions of Yen–Dollar rates from 3:35 to 8:35 and from 9:25 to 23:25 in 11/9/2001, the day of terrorism. The rates were stable till 8:35 and it became quite unstable after 9:25. The rates until 8:35 follow a slow diffusion in short time

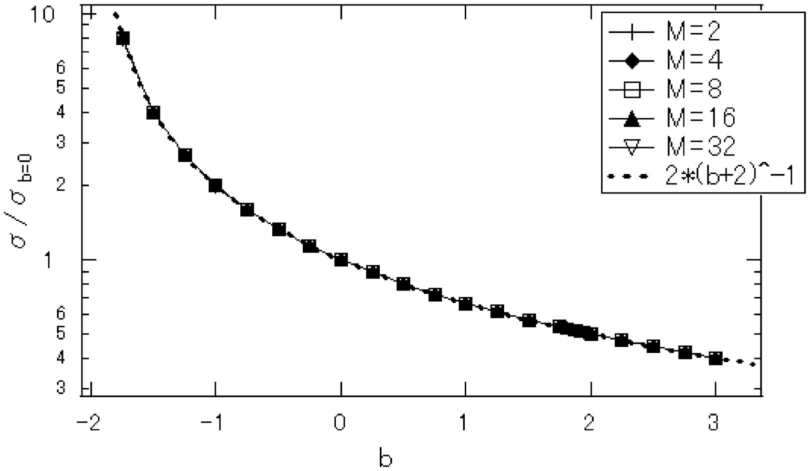

Fig. 3. Relationship between potential coefficient b and standard deviation ratio $\sigma_b(T)/\sigma_{b=0}(T)$ when the time scale $T = 10^5$. Theory (dashed line) and numerical simulations for $M = 2$ (plus), 4 (diamond), 8 (square), 16 (black triangle), 32 (white triangle) [4].

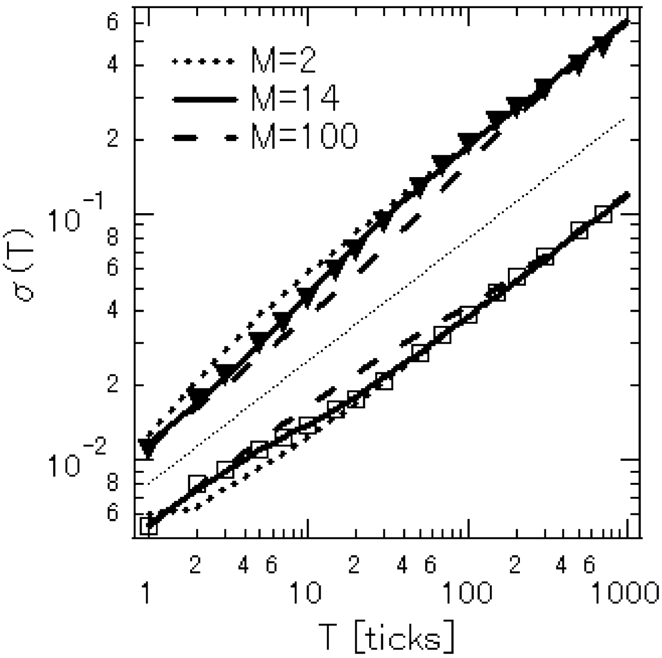

Fig. 4. Diffusion of Yen–Dollar rates. The squares and triangles show the diffusion from 3:35 to 8:35 and from 9:25 to 23:25 on 11/9/2001. The top three lines are the numerical simulation of Eq. (4) for $M = 2$, 14, 100 ticks when $b = -0.8$. The standard deviation of $f(t)$ is 0.0115 Yen/Dollar. The bottom three lines show the numerical simulation for $M = 2$, 14, 100 ticks when $b = 0.9$. The standard deviation of $f(t)$ is 0.0054 Yen/Dollar. The straight line expresses the normal diffusion with the slope 0.5.

scales, namely, the market has an attractive potential $b > 0$. As shown in Fig. 4, we can recreate the rate diffusion in all time scales by simulating Eq. (4) with $b = 0.9$ and $M = 14$ ticks. We next focus on the unstable rates after 9:25 that follow a fast diffusion in short time scales. When $b = -0.8$ and $M = 14$ ticks, the price diffusion of numerical simulation is also consistent with the rate diffusion. The price diffusion depends on M in the short time scales, although the price diffusion is independent of M in the long time scales. Therefore, if $M \neq 14$ ticks, there are gaps between the price diffusion of numerical simulations and the rate diffusion in the short time scales as shown in Fig. 4. In other real markets, we can also estimate the best value of M from such characteristics of price diffusion.

6. Discussion

We can approximate the change of market prices by the random walk in a potential field. The potential field is well approximated by a quadratic function with its center given by the moving average of past prices. The random walk process is called the PUCK model. By analyzing the model, we clarified that the statistically steady condition of price fluctuations depends on the potential coefficient b, and we also theoretically proved that the price diffusion in the long time scales is amplified $2(2+b)^{-1}$ times, independent of the size M of super moving average. In short time scales the price diffusion depends on M. We can estimate the best value of M in real financial markets by observing this dependence. We recreated the diffusion of Yen–Dollar rates in all time scales by the PUCK model. The potential coefficient b is helpful to measure the probabilistic risk of sudden change in the prices. We may be able to build better financial options that offset the risk by applying the price diffusion of the PUCK model.

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