

# Market memory and fat tail consequences in option pricing on the expOU stochastic volatility model

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## Abstract

The expOU stochastic volatility model is capable of reproducing fairly well most important statistical properties of financial markets daily data. Among them, the presence of multiple time scales in the volatility autocorrelation is perhaps the most relevant which makes appear fat tails in the return distributions. This paper wants to go further on with the expOU model we have studied in Ref. [J. Masoliver, J. Perelló, *Quant. Finance* 6 (2006) 423] by exploring an aspect of practical interest. Having as a benchmark the parameters estimated from the Dow Jones daily data, we want to compute the price for the European option. This is actually done by Monte Carlo, running a large number of simulations. Our main interest is to “see” the effects of a long-range market memory from our expOU model in its subsequent European call option. We pay attention to the effects of the existence of a broad range of time scales in the volatility. We find that a richer set of time scales brings the price of the option higher. This appears in clear contrast to the presence of memory in the price itself which makes the price of the option cheaper.

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## 1. Introduction

The model, suggested by Bachelier in 1900 as an ordinary random walk and redefined in its final version by Osborne in 1959 [1], presupposes a constant “volatility”  $\sigma$ , that is to say, a constant diffusion coefficient  $D = \sigma^2$ . However, and especially after the 1987 crash, there seems to be empirical evidence, embodied in the so-called “stylized facts”, that the assumption of constant volatility does not properly account for important features of markets [2–4]. It is not a deterministic function of time either (as might be inferred by the evidence of non-stationarity in financial time series) but a *random variable*. In its more general form one therefore may assume that the volatility  $\sigma$  is a given function of a random process  $Y(t)$ , i.e.,  $\sigma(t) = \phi(Y(t))$ .

At late eighties different stochastic volatility (SV) models (Brownian motion with random diffusion coefficient) were presented for giving a better price to the options somewhat ignoring their ability to reproduce the real price time series [5]. More recently, SV models have been suggested by some physicists as good candidates to account for the so-called stylized facts of speculative markets [6–11]. In the context of mathematical finance, we mainly have three models: the Ornstein–Uhlenbeck (OU) [9], the Heston [6] and the

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exponential OU model [12]. We have recently tried to decide which model works better [7,8]. Very recently we have studied the exponential OU SV model [12] and observed that the model shows a multiscale behavior in the volatility autocorrelation [13]. It also exhibits a leverage correlation and a probability profile for the stationary volatility which are consistent with market observations. All these features seem to make the model more complete than other SV models also based on a two-dimensional diffusion. It is worth to mention that there has been some more sophisticated models based on a three-dimensional diffusion process that reproduce the intricate set of memories in market dynamics [14,15,19]. Indeed, coming from multifractality framework, there are recent nice papers [16] with promising features to provide even a better description than the one by the common SV models [17].

## 2. The volatility model

Let us briefly summarize the main definitions and the properties of the exponential OU SV model [13]. The model consists in a two-dimensional diffusion process given by the following pair of Itô stochastic differential-equations (SDEs):

$$\dot{X}(t) = me^{Y(t)}\xi_1(t), \quad (1)$$

$$\dot{Y}(t) = -\alpha Y(t) + k\xi_2(t), \quad (2)$$

where dots represent time derivative. The variable  $X(t)$  is the undrifted log-price or zero-mean return defined as  $\dot{X} = \dot{S}(t)/S(t) - \langle \dot{S}(t)/S(t) \rangle$ , where  $S(t)$  is the asset price. The parameters  $\alpha$ ,  $m$  and  $k$  appearing in Eq. (2) are positive and non-random quantities. The two noise sources of the process are correlated Wiener processes, i.e.,  $\xi_i(t)$  are zero-mean Gaussian white noise processes with cross-correlations given by

$$\langle \xi_i(t)\xi_j(t') \rangle = \rho_{ij}\delta(t-t'), \quad (3)$$

where  $\rho_{ii} = 1$ ,  $\rho_{ij} = \rho$  ( $i \neq j$ ,  $-1 \leq \rho \leq 1$ ). In terms of the process  $Y(t)$  the volatility is given by

$$\sigma(t) = me^{Y(t)}. \quad (4)$$

It is worth to mention that multifractals models also consider a random variable which describes the logarithm of the volatility [16,17].

Among the most important results of the expOU, we must stress the following three [13]. Firstly, the stationary volatility pdf which is a log-normal distribution being quite consistent with empirical data [4]

$$p_{st}(\sigma) = \frac{1}{\sigma\sqrt{2\pi\beta}} \exp\{-\ln^2(\sigma/m)/2\beta\}, \quad (5)$$

where  $\beta = k^2/2\alpha$ . Notice that the stationary distribution broadens the tails as we increase the value of  $\beta$ . Secondly, we have the leverage effect [4]

$$\mathcal{L}(\tau) = \frac{\langle dX(t)dX(t+\tau)^2 \rangle}{\langle dX(t)^2 \rangle^2} = (2\rho k/m) \exp\{-\alpha\tau + 2\beta(e^{-\alpha\tau} - 3/4)\}H(\tau), \quad (6)$$

where  $H(\tau)$  is the Heaviside step function. One can show that

$$\mathcal{L}(\tau) \simeq \frac{2}{m} \rho k e^{\beta/2} e^{-k^2\tau} \quad (\alpha\tau \ll 1), \quad (7)$$

while leverage is negligible for large times. These approximations hold only if we take  $\beta > 1$ . Thirdly, and perhaps the most important, the volatility autocorrelation

$$\text{Corr}(\tau) = \frac{\langle dX(t)^2 dX(t+\tau)^2 \rangle - \langle dX(t)^2 \rangle^2}{\langle dX(t)^4 \rangle - \langle dX(t)^2 \rangle^2} = \frac{\exp[4\beta e^{-\alpha\tau}] - 1}{3e^{4\beta} - 1}, \quad (8)$$

which expanded in the right way allows us to observe a cascade of exponentials

$$\text{Corr}(\tau) = \frac{1}{3e^{4\beta} - 1} \sum_{n=1}^{\infty} \frac{(4\beta)^n}{n!} e^{-n\alpha\tau}. \quad (9)$$

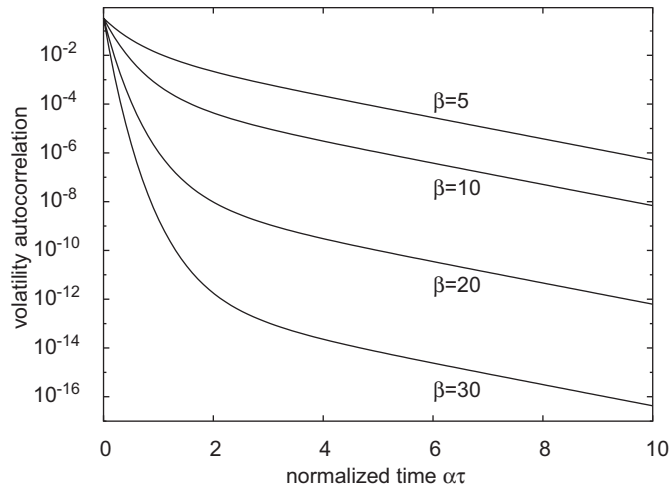


Fig. 1. Volatility autocorrelation (8) for different values of  $\beta$ . The characteristic time scale for the long time corresponds to  $1/\alpha$  while short time scale in leverage is related to  $1/k^2$ . For a large value of  $\beta$  we observe a richer multiple time scale behavior.

This expression indicates that there are infinite time scales between the limiting behaviors

$$\text{Corr}(\tau) \approx \frac{4\beta}{3e^{4\beta} - 1} e^{-\alpha\tau} \quad (\alpha\tau \gg 1), \quad \text{Corr}(\tau) = \frac{1}{3e^{4\beta} - 1} [e^{4\beta - 2k^2\tau} - 1] + O(\alpha^2 \tau^2). \quad (10)$$

As one can observe, the characteristic time scale for the long time might be associated to  $1/\alpha$  while short time scale in leverage is related to  $1/k^2$  (see Fig. 1). The distance between the smallest time scale and the largest is given by  $\beta = k^2/2\alpha > 1$ . The bigger the  $\beta$ , the larger is the distance and the richer is the cascade of multiple time scales of the process. Even more, as we enlarge the distance between smaller time scale and larger time scale, we also get a broader log-normal distribution (cf. Eq. (5)) for the volatility and a fatter distribution for the returns.

### 3. Option pricing

Having in mind latter expressions and main messages concerning  $\beta$ , we can start looking at the inferences of the volatility long-range memory in the option price. An European option is a financial instrument giving to its owner the right but not the obligation to buy (European call) or to sell (European put) a share at a fixed future date, the maturity time  $T$ , and at a certain price called exercise or striking price  $K$ . In fact, this is the most simple of a large variety of derivatives contracts. In a certain sense, options are a security for the investor thus avoiding the unpredictable consequences of operating with risky speculative stocks.

The payoff of the European call contract at maturity date  $T$  is

$$S(T) - K \quad \text{if } S(T) > K \quad \text{and } 0 \quad \text{otherwise.}$$

To compute the price of the call, we can use the average (on an equivalent martingale measure) over a large set of simulated paths which can be written mathematically as follows:

$$C_T(S) = E^*[e^{-rT}(S(T) - K)^+ | S(t=0) = S]. \quad (11)$$

To do so we assume the process defined in the pair of Eqs. (1)–(2) but with a drift equal to the risk free interest ratio  $r$  and with a market price of risk for the volatility component set to be 0. That is, keeping the current measure for the dynamics of the volatility. There are many subtleties behind the evaluation of the volatility market price of risk and we address the reader to the book by Fouque et al. [5] to get a deeper knowledge on this issue.

We thus simulate the random path assuming one day timesteps. Fig. 2 shows the different paths observed for the SV model and the log-Brownian one using  $m$  as a volatility diffusion coefficient (cf. Eq. (4)). Repeating the paths 1,000,000 times over the same time window, we can get an approximate idea of the correction to the

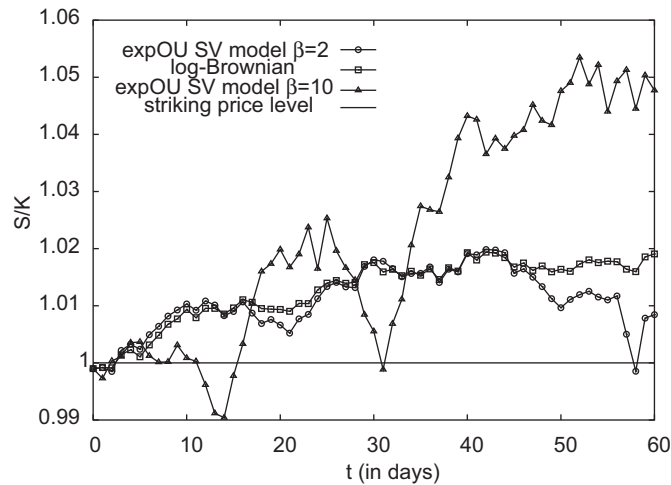


Fig. 2. Random path of the log-Brownian motion compared to a couple of expOU SV model paths with different  $\beta$ 's. Fluctuations increase wildly with  $\beta$ . Price time series take returns with a non-zero drift  $r = 1.845 \text{ day}^{-1}$  (5% annual risk free interest rate ratio). Rest of the parameters are  $k = 0.18 \text{ day}^{-1/2}$ ,  $m = 0.00141 \text{ day}^{-1/2}$  and for the log-Brownian motion (constant volatility) we take  $\sigma = m$ . All series take an initial stock price  $S = 0.999K$ .

classic Black–Scholes price formula. We recall that Black–Scholes assume a log-Brownian motion for the underlying whose price is well known and has the analytical solution

$$C_{\text{BS}}(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \quad (0 \leq t \leq T), \quad (12)$$

where  $N(z)$  is the probability integral

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-x^2/2} dx, \quad (13)$$

and its arguments are

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}, \quad d_2 = d_1 - \sigma\sqrt{T - t}. \quad (14)$$

In contrast with some contributions from mathematical finance, we are not inserting the parameters values blindly nor providing a large collection of parameters where it is quite hard to intuit the meaning and the effects of each parameter. We take the parameters set in Ref. [13] for the Dow Jones daily index as a benchmark. The parameters derived give an opposite approach to the one already performed by Fouque et al. [12] for the expOU. They are focused in analytical asymptotic results with the cases where  $\beta < 1$ , but the problem is that with this restriction, one does not have the desired cascade of time scales in the volatility autocorrelation.

We recall that  $m$  is the volatility level assuming no auxiliary diffusion process for the volatility,  $\rho$  gives the asymmetry between positive and negative return change and the appropriate sign for the leverage effect. And finally, short-range memory time scale is measured by  $k^2$  which appears to be of the order of few weeks (10–20 days). We will now focus on the effect of the largest time scale memory  $\alpha^{-1}$  which is considered to be of the order of years. We have shown in Ref. [13] that a good approximate solution for reproducing the memories of the market is performed taking  $k^2 = 1.4 \times 10^{-2} \text{ day}^{-1}$ ,  $\rho = 0.4$  and  $m = 1.5 \times 10^{-5} \text{ day}^{-1/2}$ . The comparison between shortest and largest time scale is provided with  $\beta$  and at least for the Dow Jones daily data this is around 3.8.

In Fig. 3 we plot the call price for several values of  $\beta$  averaging over the initial volatility with the log-normal distribution (5). We take maturity date at 20 days and we represent the option assuming the risk free interest ratio to be 5% per year. Even for such a small time horizon (much smaller than the volatility memory), we get important price differences. And in any case the longer the memory is, the more expensive is the option.

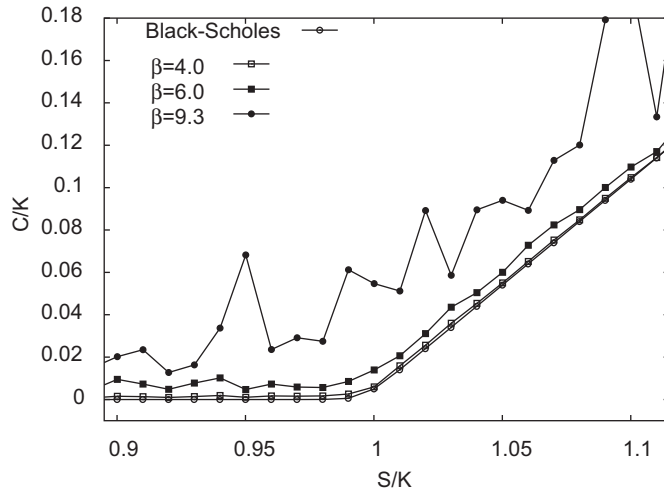


Fig. 3. The European call option in terms of the moneyness  $S/K$ . We show the Black–Scholes formula and the expOU SV model for several values of the parameters. The computation with the expOU is performed by running 1,000,000 paths and average over their final premium. Results become very noisy for  $\beta$  around 10.

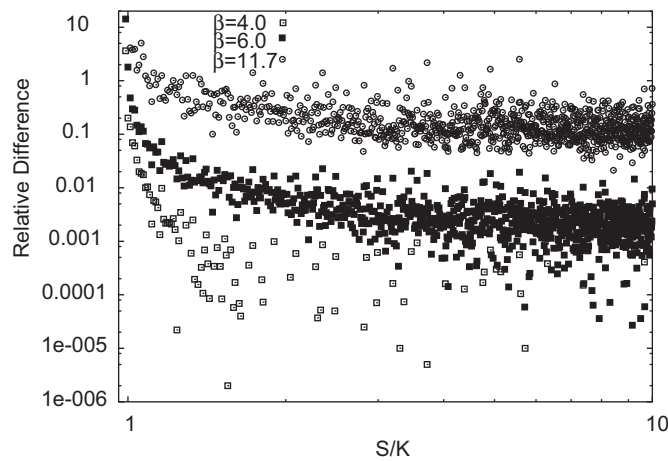


Fig. 4. The relative difference between the expOU call option and the European call option in terms of the moneyness  $S/K$ . We show the difference for several values of the parameters. The computation with the expOU is performed by running 1,000,000 paths and average over their final premium.

This can also be quantified by the relative price between new option price and the Black–Scholes one, that is,

$$\text{Relative difference} = \frac{C(S, t)}{C_{BS}(S, t)} - 1. \tag{15}$$

The results are plotted in Fig. 4. One observes that the price difference becomes more important with a monotonic increase in terms of  $\beta$ . These differences may become relatively very important for small moneyness  $S/K$ . And the decay of the relative difference for larger distances with respect to the striking price  $K$  is becoming slower with a higher value of  $\beta$ . We have tested the results with different maturity dates and with different values for  $k$ ,  $\rho$  and  $m$  with similar conclusions.

In a previous paper [18], we also have been studying the effects of the memory in the option price. In that case, however, we had memory in the price itself breaking the efficient market hypothesis. We had observed

that the call became cheaper with this kind of memory even if this is of only 1 day. The presence of memory in the volatility has opposite consequences. This paper has aimed to insist on the fact that the memory persistence in volatility affects the price, making this to be higher.

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