

Geometry of financial markets—Towards information theory model of markets

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Abstract

Most parameters used to describe states and dynamics of financial market depend on proportions of the appropriate variables rather than on their actual values. Therefore, projective geometry seems to be the correct language to describe the theater of financial activities. We suppose that the objects of interest of agents, called here baskets, form a vector space over the reals. A portfolio is defined as an equivalence class of baskets containing assets in the same proportions. Therefore portfolios form a projective space. Cross ratios, being invariants of projective maps, form key structures in the proposed model. Quotation with respect to an asset \mathcal{E} (i.e. in units of \mathcal{E}) is given by linear maps. Among various types of metrics that have financial interpretation, the min–max metric on the space of quotations can be introduced. This metric has an interesting interpretation in terms of rates of return. It can be generalized so that to incorporate a new numerical parameter (called temperature) that describes agent's lack of knowledge about the state of the market. In a dual way, a metric on the space of market quotation is defined. In addition, one can define an interesting metric structure on the space of portfolios/quotation that is invariant with respect to hyperbolic (Lorentz) symmetries of the space of portfolios. The introduced formalism opens new interesting and possibly fruitful fields of research.

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1. Introduction

In a majority of the models considered in economics one cannot ask questions about symmetries of the considered phenomena, especially if one puts the stress on group theoretical aspects. The reason is that one can hardly speak about invariance (covariance) of terms used in analysis or numerical values returned by most models [1]. We would like to argue that projective geometry, equipped with an appropriate metric structure and some measure of investors performance, might form a precise formalism that allows us to carry out objective (quantitative) analysis of investment processes and symmetries of their market context. We describe a simple geometrical model of a financial market—we call it information theory model of markets (ITMM)—that explores ideas of projective geometry. Our model presents in some sense a picture of financial markets

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dual to that assumed in the most popular ones, Capital asset pricing model and arbitrage pricing models [2]. Investors, due to their lack of knowledge, wrong prognosis for the future or simple fear, behave in an unpredictable, chaotic way. Prices are determined by their decisions—in the same way as the gas pressure is determined by (chaotic) particles dynamics. A non-random pricing of capital assets follows from investors knowledge and possible random factors cancel themselves due to variety of strategies adopted by investors if the market is liquid enough. The formalism of projective geometry allows us to carry out an analysis of invariant and covariant quantities. A detailed axiomatic formulation of the model will be given elsewhere [3], here we would like to present only some basic features. The paper is organized as follows. In the next section we give some basic definitions and describe the mathematical tools we are going to use. Then we show the importance of metric structures and give two exemplary metrics. It follows that some important analogies with physical theories can be expected. Finally, we discuss a possible connection between investors performance and knowledge about markets measured by means of information theory.

2. Projective geometry as a formalism describing investments

The market determines what goods are made and what products are bought and sold. We assume that objects of investors interest span a $(N + 1)$ -dimensional vector space G over the reals. Elements of this vector space are called *baskets*. Let us fix some basis $\{\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_N\}$ in G . $\mathbf{g}_\mu \in G$, the μ th element of the basis, is called the μ th *asset* (market good). Assets, although selected in an arbitrary way, are distinguished because they are used for effective bookkeeping, accounting, market analysis and so on. For any basket $p \in G$ we have a unique representation

$$p = \sum_{\mu=0}^N p_\mu \mathbf{g}_\mu.$$

The coefficient $p_\mu \in \mathbb{R}$ is called the μ th *coordinate of the basket*. A *portfolio* is defined as an equivalence class of non-empty baskets (that is in $G \setminus \{0\}$) [4]. Two baskets p' and p'' are equivalent if and only if there exists $\lambda \in \mathbb{R}$, such that

$$\sum_{\mu=0}^N p'_\mu \mathbf{g}_\mu = \sum_{\mu=0}^N \lambda p''_\mu \mathbf{g}_\mu.$$

Equivalently,

$$(p'_0, \dots, p'_N) = (\lambda p''_0, \dots, \lambda p''_N).$$

If for a given portfolio we have $p_\mu \neq 0$, then there exists such a basket representing this portfolio that it contains exactly one unit of asset \mathbf{g}_μ . Coordinates of this basket, $p = (p_0, \dots, p_{\mu-1}, 1, p_{\mu+1}, \dots, p_N)$, are called *inhomogeneous coordinates* of the portfolio p with respect to μ th asset. If $p_\mu = 0$, $p = (p_0, \dots, p_{\mu-1}, 0, p_{\mu+1}, \dots, p_N)$, then we say that the portfolio p is *improper* for the μ th asset. *Market quotation* U in units of v th asset is a linear map $U(\mathbf{g}_v, \cdot) : G \rightarrow \mathbb{R}$. The map U associates with a given portfolio p its current value in units of \mathbf{g}_v :

$$(Up)_v = U(\mathbf{g}_v, p) = \sum_{\mu=0}^N U(\mathbf{g}_v, \mathbf{g}_\mu) p_\mu, \tag{1}$$

where $U(\mathbf{g}_v, \mathbf{g}_\mu)$ is the price of a unit of μ th asset given in units of v th asset.

2.1. Basic definition and ideas

We require that

$$U(\mathbf{g}_\mu, U(\mathbf{g}_v, p)\mathbf{g}_v)\mathbf{g}_\mu = U(\mathbf{g}_\mu, p)\mathbf{g}_\mu$$

for p and \mathbf{g}_μ and \mathbf{g}_ν being exchangeable assets (that is $U(\mathbf{g}_\mu, \mathbf{g}_\nu) \neq 0$ and $U(\mathbf{g}_\mu, \mathbf{g}_\nu) \neq \pm \infty$, so inserting $p = \mathbf{g}_\rho$ we get

$$U(\mathbf{g}_\mu, \mathbf{g}_\nu)U(\mathbf{g}_\nu, \mathbf{g}_\rho) = U(\mathbf{g}_\mu, \mathbf{g}_\rho) \tag{2}$$

for any μ, ν, ρ (this constraint is often called the triangle rule). Therefore quotations are transitive [5]. If we set $\mu = \nu = \rho$ in Eq. (2) then we see that there are two possibilities $U(\mathbf{g}_\mu, \mathbf{g}_\mu) = 1$ or $U(\mathbf{g}_\mu, \mathbf{g}_\mu) = 0$. The case $U(\mathbf{g}_\mu, \mathbf{g}_\mu) = 1$ implies projectivity of U : $U((Up)_\mu, \mathbf{g}_\mu)_\mu = (Up)_\mu$. The case $U(\mathbf{g}_\mu, \mathbf{g}_\mu) = 0$ means that the μ th asset is not subjected to quotation in the market (one can only, for example, present somebody with such an asset). For $\mu = \rho$ we get $U(\mathbf{g}_\mu, \mathbf{g}_\mu) = 1$ and therefore

$$U(\mathbf{g}_\mu, \mathbf{g}_\nu) = (U(\mathbf{g}_\nu, \mathbf{g}_\mu))^{-1}.$$

In general, the quotation map can be represented by a $(N + 1) \times (N + 1)$ matrix with (μ, ν) -th entry given by $U_{\mu\nu} := U(\mathbf{g}_\mu, \mathbf{g}_\nu)$. The simplest way of determining this matrix consists in selecting some asset that is called *the currency*. Suppose that the asset \mathbf{g}_0 is selected as the currency. The matrix $U_{\mu\nu}$ is defined uniquely by N values $u_k := U(\mathbf{g}_0, \mathbf{g}_k)$ for $k = 1, \dots, N$. (Note that $U_{00} = 1$). If $u_0 := 1$, due to the transitivity (Eq. (2)) all entries of $(U_{\mu\nu})$ are determined by the formula

$$U_{\mu\nu} = u_\mu^{-1}u_\nu. \tag{3}$$

After this Eq. (1) simplifies to

$$(Up)_\nu = \sum_{\mu=0}^N u_\mu p_\mu u_\nu^{-1}.$$

We can extend the domain of applicability of Eq. (3) to include the case $u_k = 0$ by setting $u_k^{-1} := 0$. Sometimes we have to rescale the prices u_k in units proportional to \mathbf{g}_0 , (e.g. if \mathbf{g}_0 represents shares, after split, after currency denomination and so on). Therefore it is convenient to identify quotations $U = (\lambda, \lambda u_1, \dots, \lambda u_N)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ that introduces homogeneous coordinates. We say that the portfolio p is *balanced* for the quotation U if there is such an asset \mathbf{g}_μ , so that the value of p in units of \mathbf{g}_μ is 0 [6], that is,

$$(Up)_\mu = \sum_{\nu=0}^N U(\mathbf{g}_\mu, \mathbf{g}_\nu)p_\nu = \sum_{\nu=0}^N u_\nu p_\nu u_\mu^{-1} = 0.$$

For quotation denominated in currency this formula simplifies to $\sum_{\mu=0}^N u_\mu p_\mu = 0$. The linearity of these equations allows of the simple interpretation: portfolio p is balanced if the corresponding point belongs to the hyperplane representing quotation U (Fig. 1, Table 1).

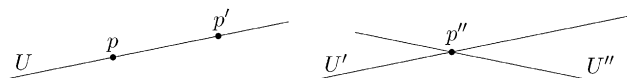


Fig. 1. Two different portfolios p, p' balanced for the same quotation U and a portfolio p'' balanced for two different quotations U', U'' (Duality!).

Table 1
Projective geometry dictionary

	Market	Projective geometry
p	portfolio	point
U	quotation	hyperplane
$Up = 0$	portfolio is balanced for U	point p lies in quotation hyperplane

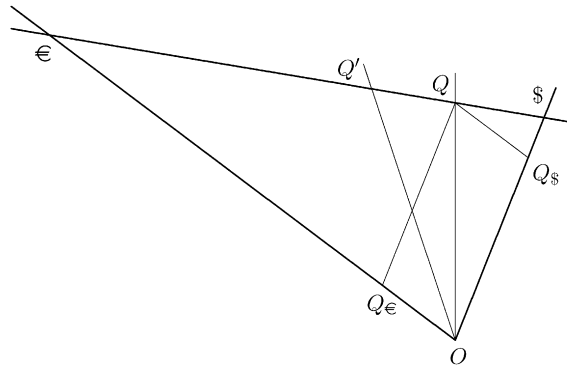


Fig. 2. Exchange ratios.

An important invariant can be defined in projective geometry—a cross ratio of four points [7]. For exchange ratios it describes the relative change of quotation (cf. Fig. 2):

$$\{\$, Q, Q', \epsilon\} := \frac{c'_{\$ \rightarrow \epsilon}}{c_{\$ \rightarrow \epsilon}} = \frac{q'_\$ q_\epsilon}{q_\$ q_\epsilon} = \frac{|Q'\epsilon| |Q\$|}{|Q'\$| |Q\epsilon|} = \frac{P(\Delta_{Q'\epsilon O}) P(\Delta_{Q\$ O})}{P(\Delta_{Q'\$ O}) P(\Delta_{Q\epsilon O})},$$

where $c_{\$ \rightarrow \epsilon} := q_\$/q_\epsilon$ is the exchange ratio $\$ \rightarrow \epsilon$ (one obtains for $q_\$$ dollar q_ϵ euro) etc. and $P(\Delta_{ABC})$ denotes the area of the triangle with vertices A, B , and C . In Fig. 2 lengths of the segments $Q\$$ and $Q\epsilon$ are proportional to $q_\$$ and q_ϵ , respectively. The invariance of cross ratios is crucial to our model.

2.2. Example: trading in a single asset

Let us consider the cross ratio $[\mathfrak{G}, U_{\rightarrow \mathfrak{G}}, U_{\mathfrak{G} \rightarrow}, \$]$ for $U_{\rightarrow \mathfrak{G}} := (v, v e^{p_{\rightarrow \mathfrak{G}}}, \dots)$ and $U_{\mathfrak{G} \rightarrow} := (w, w e^{p_{\mathfrak{G} \rightarrow}}, \dots)$ and the points \mathfrak{G} and $\$$ given by crossing of the prime line $U_{\rightarrow \mathfrak{G}} U_{\mathfrak{G} \rightarrow}$ and one-asset portfolios hyperplanes \mathfrak{G} and $\$$ corresponding to assets \mathfrak{G} and $\$$. $p_{\rightarrow \mathfrak{G}}$ and $p_{\mathfrak{G} \rightarrow}$ are the logarithmic quotations for buying and selling, respectively, and the dots (...) represent quotations for the remaining assets and need not be the same for both quotations. The logarithm of the cross ratio $[\mathfrak{G}, U_{\rightarrow \mathfrak{G}}, U_{\mathfrak{G} \rightarrow}, \$]$ on the straight line $U_{\rightarrow \mathfrak{G}} U_{\mathfrak{G} \rightarrow}$ is equal to:

$$\ln[\mathfrak{G}, U_{\rightarrow \mathfrak{G}}, U_{\mathfrak{G} \rightarrow}, \$] = \ln \left[\frac{w e^{p_{\mathfrak{G} \rightarrow}}}{w e^{p_{\mathfrak{G} \rightarrow}} - v e^{p_{\rightarrow \mathfrak{G}}}}, 1, 0, \frac{w}{w - v} \right] = \ln \frac{v w e^{p_{\mathfrak{G} \rightarrow}}}{v e^{p_{\rightarrow \mathfrak{G}}} w} = p_{\mathfrak{G} \rightarrow} - p_{\rightarrow \mathfrak{G}}.$$

In that way we arrive at a geometrical interpretation of the logarithmic rate of return $p_{\mathfrak{G} \rightarrow} - p_{\rightarrow \mathfrak{G}}$ which turns out to be an invariant of the space of portfolios. Therefore, besides being a convenient accountancy tool, it forms an invariant characterization of transactions.

3. Metric structures

It is common lore that price movements are best described by diffusion processes. Diffusion equations of various types involve the Laplace operator and therefore metric structure. Metric structures are to some extent independent of the configuration (phase) space structure. One of our aims is to find a suitable metrics on the projective space. Various premises rooted in finance theory can be used to select a metric structure on the space of portfolios. For example, often we would like to know which market movements are equivalent to portfolio modifications. Below we describe two classes of metrics that we were able to construct in an explicit way. Both have interesting physical connotations. There probably is quite a lot of other interesting metrics yet to be found.

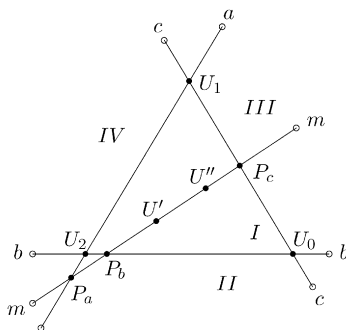


Fig. 3. Quotation in a three-assets market ($\mathbb{R}P^2$).

3.1. Exemplary metric structure

Let us try to define a metrics on the space of quotations. Any two different quotations U' and U'' determine a projective prime line. To define a cross ratio we need two additional points lying in that line. It seems natural to select them, let us consider two hyperplanes of improper quotations for two basic assets. These hyperplanes cut the projective space $\mathbb{R}P^N$ into 2^N N -dimensional simplexes. Suppose that the quotations belong to the same simplex—only then the distance would be finite. Each hyperplane of improper quotation for an asset g_μ is cut by the prime line. In this way we select $N + 1$ points but only two of them, say P_b and P_c , lie in the vicinity of U' and U'' —and only these two points belong to the boundary of the simplex that contains U' and U'' , cf. Fig. 3. The cross ratio $[P_b, U', U'', P_c]$ can be used to define the distance (metrics):

$$d(U', U'') = \ln([P_b, U', U'', P_c]) = \ln \frac{|U'P_c| |U''P_b|}{|U'P_b| |U''P_c|}, \tag{4}$$

where $|P_1P_2|$ denotes euclidean distance of points P_1 and P_2 . After some tedious but elementary calculations the metrics can be given in a more transparent form:

$$\begin{aligned} d(U', U'') &= \ln([P_b, U', U'', P_c]) = \ln \frac{|U'P_c| |U''P_b|}{|U'P_b| |U''P_c|} = \ln \left(\max_\mu \left(\frac{u''_\mu}{u'_\mu} \right) \right) - \ln \left(\min_\mu \left(\frac{u''_\mu}{u'_\mu} \right) \right) \\ &= \max_\mu (r_\mu(U', U'')) - \min_\mu (r_\mu(U', U'')) = \max_\mu (r_\mu(U', U'')) + \max_\mu (r_\mu(U'', U')). \end{aligned} \tag{5}$$

The function $r_\mu(U', U'')$ is known in finance as *the interval interest rate*. We have already proposed a method that allows us to measure quantitatively investors qualifications [8]. Inspired by previous results and statistical physics, we can introduce a temperature-like parameter $T \in (0, \infty)$ in the metrics given by Eq. (5). Such a generalized metrics (actually, a family of mean rates for Gibbs ensembles of rational transactions with incomplete information Ref. [7]) takes the following form:

$$d(U', U''; T) := \frac{\sum_\mu r_\mu(U', U'') e^{r_\mu(U', U'')/T}}{\sum_\mu e^{r_\mu(U', U'')/T}} + \frac{\sum_\mu r_\mu(U'', U') e^{r_\mu(U'', U')/T}}{\sum_\mu e^{r_\mu(U'', U')/T}}.$$

It should be possible to define canonical ensembles of portfolios, the temperature (entropy) of portfolios and, possibly, various thermodynamics-like potentials in a way analogous to that of Ref. [7].

3.2. Hyperbolic (Lorentz) geometry

We were able to identify another interesting metrics. Consider quotations at two different times t' and t'' in a simplified, two-assets market. Let the homogeneous coordinates are $\hat{p}^{*'} = (g\hat{p}'_0, g\hat{p}'_1)$ and $\hat{p}^{*''} = (g\hat{p}''_0, g\hat{p}''_1)$,

respectively. Suppose the quotations are not equal, $\hat{p}^{*'} \neq \hat{p}^{*''}$. The linear transformation

$$\hat{S} = \hat{S}(\hat{p}^{*'}, \hat{p}^{*''}) := \frac{1}{g\hat{p}'_0 g\hat{p}''_1 - g\hat{p}''_0 g\hat{p}'_1} \begin{pmatrix} -g\hat{p}'_1 + g\hat{p}''_1 & g\hat{p}'_0 - g\hat{p}''_0 \\ -g\hat{p}'_1 - g\hat{p}''_1 & g\hat{p}'_0 + g\hat{p}''_0 \end{pmatrix}$$

changes the basis in such a way that the quotations $\hat{p}^{*'}$ and $\hat{p}^{*''}$ have coordinates ${}_t\hat{p}' := (1, -1)$ and ${}_t\hat{p}'' := (1, 1)$. From the physicist point of view, the directions $(1, -1)$ and $(1, 1)$ define the propagation of light in a two-dimensional spacetime but the “time” coordinate measures now the quantity of a good in the basket in question and therefore does not have any causal structure context. We can accept this directions as absolute directions (“light cone”). The underlying metric structure can also be found. In the dual representation, that is in the space of portfolios, two portfolios balanced on quotations $\hat{p}^{*'}$ and $\hat{p}^{*''}$ are infinitely separated. Explicit form of the metrics on the space of portfolios is as follows:

$$d(p^{*'}, p^{*''}) = |\arctan v(p^{*'}) - \arctan v(p^{*''})|,$$

where

$$v(p^*) = v(p^*, \hat{p}^{*'}, \hat{p}^{*''}) = \frac{g\hat{p}''_0(g\hat{p}''_0 - g\hat{p}'_0) + g\hat{p}'_1(g\hat{p}''_1 - g\hat{p}'_1)}{g\hat{p}'_0(g\hat{p}''_0 + g\hat{p}'_0) + g\hat{p}'_1(g\hat{p}''_1 + g\hat{p}'_1)}.$$

Note that if we neglect details of the economic processes that make capital then one can change the content of a portfolio only if one “travels with speed of light” in the market. The trajectory of a massive particle is in the additive velocity parameter space infinitely separated from the light cone. Analogously, a portfolio with quotation different from zero is infinitely separated from the hyperspace of portfolios balanced on the actual quotation (that is worthless) [9].

3.3. Information theory context

The projective geometry structure of clear-cut market model with a metrics that respects symmetries of the modelled processes should yet be completed by discussion/construction of algorithms that governs the supply and demand aspects of agents behavior. These algorithms should be optimal from the metrical structure point of view and, of course, respect specific regulations laid down by authorities. For example, in the simple merchandising mathematician model [10] and Kelly optimal bets [11] the optimal market strategies have direct connections with the Boltzmann/Shannon entropy. These examples suggest that there might be a unified description of market phenomena that involves tools from geometry, statistical physics and information theory. And the key ingredients would probably follow from the underlying metric structure.

4. Conclusions: towards information theoretical description of markets

We have attempted at a formulation of kind of a market symmetry principle: *conclusions drawn from a logically complete market model are invariant or covariant with respect to projective symmetry transformations.* We anticipate that metric structures might play a key role that would pave the way for information theoretical description of market phenomena. This point of view is supported by the explicit examples given in the paper. The presented projective geometry formalism although simplified, is, to the best of our knowledge, the only one that attempts to introduce metric structure to finance theory models that respect observed market processes symmetries, e.g. preselected absolute directions [12]. Discovered in the XIXth century richness of projective geometry ideas can be used to investigate scale-free invariants of market transactions. And vice versa: we envisage that identified by the practitioners of finance invariant and covariant quantities would get new interesting interpretations and therefore gain importance [13]. Especially promising is the exploration of dualities—for example, various clustering phenomena discovered in currency networks (FOREX) [14] should have a dual representation. Discrete metric problems on graphs, e.g. connected with the shortest path strategies concern spaces with radically different topologies and their projective geometry formulation are beyond the scope of the present paper. Nevertheless they are very important and deserve a separate analysis,

cf. Ref. [15]. Another interesting class problem that deserves a separate analysis is related to the interest rate parity theorem and the involved symmetrie [16].

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- [1] An invariant of a process (phenomenon, transformation, etc.) is a numerical parameter whose value remains constant during that process. Analogously, a covariant is a parameter whose numerical value has only in a (relative) sense a preselected coordinate system but changes in a specified way if the coordinate system is changed (e.g. given by some symmetry transformation).
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- [5] Note that this means that we treat all taxes, brokerages, etc. as liabilities and therefore as separate assets. In this way we are not forced to consider friction in the model. Models that have scale effects (projective symmetry is broken) should have a dual description in terms of non-transitive quotations.
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