

# A mechanism to derive multi-power law functions: An application in the econophysics framework

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Available online 1 March 2007

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## Abstract

It is generally recognized that economical systems, and more in general complex systems, are characterized by power law distributions. Sometime, these distributions show a changing of the slope in the tail so that, more appropriately, they show a multi-power law behavior. We present a method to derive analytically a two-power law distribution starting from a single power law function recently obtained, in the frameworks of the generalized statistical mechanics based on the Sharma–Taneja–Mittal information measure. In order to test the method, we fit the cumulative distribution of personal income and gross domestic production of several countries, obtaining a good agreement for a wide range of data.

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*Keywords:* Two-power law distribution; Sharma–Taneja–Mittal information measure; Distribution of personal income and gross domestic production

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## 1. Introduction

Free-scale behavior in the economical systems have been observed since 19th century, when Pareto noticed that the cumulative distribution of the personal income  $P(x) = \int_x^\infty p(y) dy$  of several countries behaves like a power law function. Afterwards, Gibrat clarified that such a power law behavior holds only for the high income region, whilst in the low-middle income region, which includes almost the whole body of data, the curve is well fitted by a log-normal distribution.

Actually, the problem concerning the real profile showed by the function  $P(x)$  in the whole range of the accessible data is still an open question. In particular, it has been suggested [1] that deformed exponential functions derived recently in the field of the generalized statistical mechanics, can be fruitfully employed to modeling analytically the cumulative distribution  $P(x)$  for a wide range of the income values.

Notwithstanding, the recent analysis based on a huge quantity of data nowadays accessible, shown that sometime the crossover among the low-middle region (the log-normal region) and the high region in the upper tail of the distribution (the Pareto region, with a power law behavior  $P(x) \sim x^{-s}$ , where  $s$  is a positive constants quite generally  $1 \leq s \leq 2$ ), does not occur smoothly, giving origin to *knee or ankle effects* (see for instance

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Ref. [2]). Moreover, in some cases, it has been observed a deviation from the Pareto behavior in the highest region, which can originate a new power law behavior  $P(x) \sim x^{-\tilde{s}}$  with a different slope  $\tilde{s} \neq s$ .

The complicate profile in the shape of  $P(x)$  cannot be accounted for by a generalized exponential with a single power law behavior. This open the questions: how can we describe the shape observed in  $P(x)$  with an analytically simple function?

In the present contribution, we introduce a mechanism which permits to generate multi-power law functions by employing deformed exponentials and logarithms with a single power law asymptotic behavior.

Notice that, two-power law behavior have been observed also in various economical systems like, for instance, in the cumulative distribution of the personal income [3], in the cumulative distribution of the land price [4] or in the returns of many market indexes [5]. On a general ground, two-power law behavior have been noticed also in different physics fields as well as in biological, geological and social sciences. Among the many, we quote the dielectric relaxation [6], the re-association in folder proteins [7], and others [8].

It is worthy to remark that there have been proposed different methods in literature [7,9,10] to produce generalized distributions with a double-power law behavior which differ from the one advanced in the following.

## 2. Deformed logarithms and exponentials

Generalized exponential functions  $\mathcal{E}(x)$ , interpolating between the standard exponential  $\exp(x)$  for  $x \ll 1$  and the power law  $x^{-s}$  for  $x \gg 1$ , arise naturally in the study of thermostatic properties of complex systems which show free-scale feature. In Ref. [11], it has been postulated a very general expression for the entropy of such a system

$$S(p) = - \int p(x) A(p(x)) dx, \tag{2.1}$$

(in the unity of Boltzmann constant  $k_B = 1$ ), where  $A(x)$  plays the role of a generalized logarithm, the inverse function of  $\mathcal{E}(x)$ . By requiring that the distribution, derivable through a variational problem, assumes the form

$$p(x) = \mathcal{E} \left( - \sum_{j=1}^M \beta_j |x|^{\mu_j} \right), \tag{2.2}$$

which mimics the well-known Boltzmann–Gibbs distribution, the following functional equation has been obtained:

$$\frac{d}{dx} [x A(x)] = \lambda A \left( \frac{x}{\alpha} \right). \tag{2.3}$$

Here,  $\alpha$  and  $\lambda$  are constants given by

$$\alpha = \left| \frac{1+r-\kappa}{1+r+\kappa} \right|^{1/2\kappa}, \quad \lambda = \frac{|1+r-\kappa|^{(r+\kappa)/2\kappa}}{|1+r+\kappa|^{(r-\kappa)/2\kappa}}. \tag{2.4}$$

The quantities  $\beta_j$  in Eq.(2.2) play the role of Lagrange multipliers associated to the  $M$  constraints  $\int |x|^{\mu_j} p(x) dx = \mathcal{O}_j$  which represent the  $\mu_j$ th momenta of  $x$ . Typically, the constants  $\mu_j$  are integers (for instance,  $\mu_1 = 0$  gives the normalization  $\int p(x) dx = \mathcal{O}_1$ ,  $\mu_2 = 1$  is the mean value  $\langle x \rangle = \mathcal{O}_2$ ) but for sake of generality we assume  $\mu_j \in \mathbb{R}$ .

The most general solution of Eq. (2.3), accounting for the boundary conditions  $A(1) = 0$  and  $(d/dx)A(x)|_{x=1} = 1$  derived from certain physically and mathematically justified assumptions, is given by

$$A(x) \equiv \ln_{\{\kappa,r\}}(x) = x^r \frac{x^\kappa - x^{-\kappa}}{2\kappa}, \tag{2.5}$$

which recover the standard logarithm in the  $(\kappa, r) \rightarrow (0, 0)$  limit.

By requiring that  $\ln_{\{\kappa,r\}}(x)$  is a continuous, monotonic, normalizable, concave and increasing function for  $x \in (0, +\infty)$ , we obtain the restrictions  $-\kappa \leq r \leq \kappa$ , if  $0 \leq \kappa < 1/2$  and  $|\kappa| - 1 \leq r \leq 1 - |\kappa|$ , if  $1/2 \leq \kappa < 1$ .

Notwithstanding, for particular applications some of the above mathematical requirements can be relaxed permitting less restrictive conditions for the deformation parameters. For instance, in certain practical situations one is welling with a normalization in a finite interval  $x \in (0, x_{\max})$  [12] and we can discard the condition  $|\kappa| < 1$ .

In the following, we require only that Eq. (2.5) be a monotonic function, so that its inverse function, the generalized exponential  $\exp_{\{\kappa,r\}}(x)$ , certainly exists. This is accomplished by requiring only that  $-|\kappa| < r < |\kappa|$ .

From Eq. (2.5) we obtain that  $\ln_{\{\kappa,r\}}(x) \rightarrow x^{r+|\kappa|}/|2\kappa|$  for  $x \rightarrow +\infty$  and  $\ln_{\{\kappa,r\}}(x) \rightarrow -x^{r-|\kappa|}/|2\kappa|$  for  $x \rightarrow 0$ , whilst  $\ln_{\{\kappa,r\}}(x) \rightarrow (x - 1)$  for  $|x - 1| \ll 1$ . In the same way, we have that  $\exp_{\{\kappa,r\}}(x) \rightarrow |2\kappa x|^{1/(r \pm |\kappa|)}$  for  $x \rightarrow \pm\infty$ , whilst  $\exp_{\{\kappa,r\}}(x) \rightarrow 1 + x$  for  $x \rightarrow 0$ . Thus, the deformed exponential  $\exp_{\{\kappa,r\}}(x)$  interpolates with continuity between the standard exponential  $\exp(x) \simeq 1 + x$ , for  $x \rightarrow 0$ , and the power law  $|x|^{-s}$  with slope  $s = -1/(r \pm |\kappa|)$ , for  $x \rightarrow \pm\infty$ .

Finally, accounting for the solution (2.5), the entropy (2.1) assumes the form

$$S_{\kappa,r}(p) = - \int p(x) \ln_{\{\kappa,r\}}(p(x)) dx, \tag{2.6}$$

which recovers, in the limit  $(\kappa, r) \rightarrow (0, 0)$ , the Shannon–Boltzmann–Gibbs entropy  $S = - \int p(x) \ln p(x) dx$ . This entropic form, introduced previously in literature in Refs. [13–15], is known as the Sharma–Taneja–Mittal information measure and has been applied recently in the formulation of a possible thermostatistics theory [16,17].

### 3. Two-power law function

Endowed with the deformed logarithm  $\ln_{\{\kappa,r\}}(x)$  and the deformed exponential  $\exp_{\{\kappa,r\}}(x)$  we can construct the quantity

$$\Pi_{\sigma_1}(x) = \exp_{\{\kappa_1,r_1\}}(a_1 \ln_{\{\kappa_1,r_1\}}(x)), \tag{3.1}$$

where  $\sigma_1$  denotes the set of parameters  $\sigma_1 \equiv (\kappa_1, r_1, a_1)$ , with  $a_1 \geq 1$ . The function (3.1) is therefore employed in the following construction:

$$f(x) = \Pi_{\sigma_1} \circ \exp_{\{\kappa_2,r_2\}}(x) \equiv \Pi_{\sigma_1}(\exp_{\{\kappa_2,r_2\}}(-x)). \tag{3.2}$$

We observe that, for  $a_1 = 1$  expression (3.2) reduces to  $\exp_{\{\kappa_2,r_2\}}(-x)$ , for  $(\kappa_1, r_1) \rightarrow (0, 0)$  we obtain  $[\exp_{\{\kappa_2,r_2\}}(-x)]^{a_1}$ , whilst for  $(\kappa_1, r_1) = (\kappa_2, r_2)$  we obtain  $\exp_{\{\kappa_1,r_1\}}(-a_1 x)$ .

Accounting for the asymptotic behavior of the deformed exponential and logarithm we can distinguish three regions in the range  $x > 0$  of  $f(x)$ . A first region, for  $a_1 x \ll 1$ , characterized by the linear behavior

$$f(x) \sim 1 - a_1 x, \tag{3.3}$$

like the exponential  $\exp(-a_1 x)$  does for  $x \rightarrow 0$ . A second intermediate region, for  $x \ll 1 \ll a_1 x$ , where  $f(x)$  is characterized by the power law behavior

$$f(x) \sim x^{-s_1}, \tag{3.4}$$

with slope  $s_1 = 1/(|\kappa_1| - r_1)$ .

Finally, for  $a_1 x \gg 1$  we obtain the asymptotic power law behavior

$$f(x) \sim x^{-s_2}, \tag{3.5}$$

whose slope is now  $s_2 = 1/(|\kappa_2| - r_2)$ .

Thus,  $f(x)$  behaves like a power law function both in the middle and in the far region of  $x > 0$  with slopes  $s_1$  and  $s_2$ , respectively. In this sense, we call Eq. (3.2) a two-power law function.

Form the above analysis, we easily realize that the constant  $a_1$ , introduced in the definition of  $\Pi_{\sigma_1}(x)$ , gives approximatively the width of the intermediate region having slope  $s_1$ .

As an example, let us specialize Eq. (3.2) to the case  $r = 0$ . In this situation, the generalized exponential and logarithm assume, respectively, the expression [20]

$$\exp_{\{\kappa\}}(x) = (\kappa x + \sqrt{1 + \kappa^2 x^2})^{1/\kappa}, \tag{3.6}$$

and

$$\ln_{\{\kappa\}}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}. \tag{3.7}$$

In Fig. 1, we plot the function

$$P(x) = \exp_{\{\kappa_1\}}(a_1 \ln_{\{\kappa_1\}}(\exp_{\{\kappa_2\}}(-x))), \tag{3.8}$$

for the values  $\kappa_1 = 0.35$ ,  $\kappa_2 = 1.4$  and  $a_1 = 10^3$ . In the same graphic, the dot-dashed line depicts the cumulative integral  $I(x) = \int_x^\infty p(y) dy$  of the log-normal distribution

$$p(x) = \frac{1}{(2\pi)^{1/2} x} \exp\left(-\frac{1}{2} \ln^2 x\right). \tag{3.9}$$

The dashed lines represent the asymptotic prolongation of the power law behavior of  $P(x)$  whose slopes are given, respectively, by  $s_1 = 1/\kappa_1 = 2.85$  and  $s_2 = 1/\kappa_2 = 0.71$ . We observe a good agreement between the functions  $I(x)$  and  $P(x)$  only in the low region of  $x$ .

#### 4. Application to econophysics

In the following, we employ the function derived in the previous Section 3 to fit some distributions data obtained in the economy framework.

We pose  $P(x) = f(-\beta |x|^\mu)$ , with  $\beta$  and  $\mu$  fitting parameters, the cumulative distribution representing the probability of finding a value  $X$  equal to, or greater than  $x$ .

In Fig. 2, we present the results of the fit (in log–log scale) for the data of the inverse cumulative distribution of the personal income of Japan (1975) obtained in Ref. [18] and USA (2000) obtained in Ref. [3], as well as, the data of the inverse cumulative distribution of the gross domestic production of Brazil (1996) and Germany (1998) obtained in Ref. [3].

In every graphic, we report the dashed lines representing the asymptotic behavior of  $P(x)$  in the two-power law regions with slope given by  $s_1 = \mu/\kappa_1$  and  $s_2 = \mu/\kappa_2$ , respectively.

The data fit are reported in Table 1.

The crossover between the first and the second power law region, causing a reduction of the slope, with  $s_2 < s_1$  (UK, Brazil and USA), is named *kink effect* [3]. Similarly, the crossover between the first and the second power law region causing an increase of the slope, with  $s_2 > s_1$  (Japan), is named *ankle effect*.

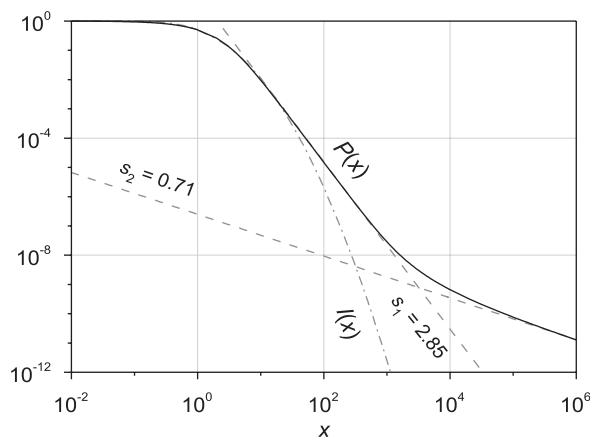


Fig. 1. Log–log plot of the two-power law function (3.8) (solid line). The dot-dashed line is the cumulative integral of the log-normal function. The dashed lines denote the asymptotic extension of the function  $P(x)$  in the two-power law regions.

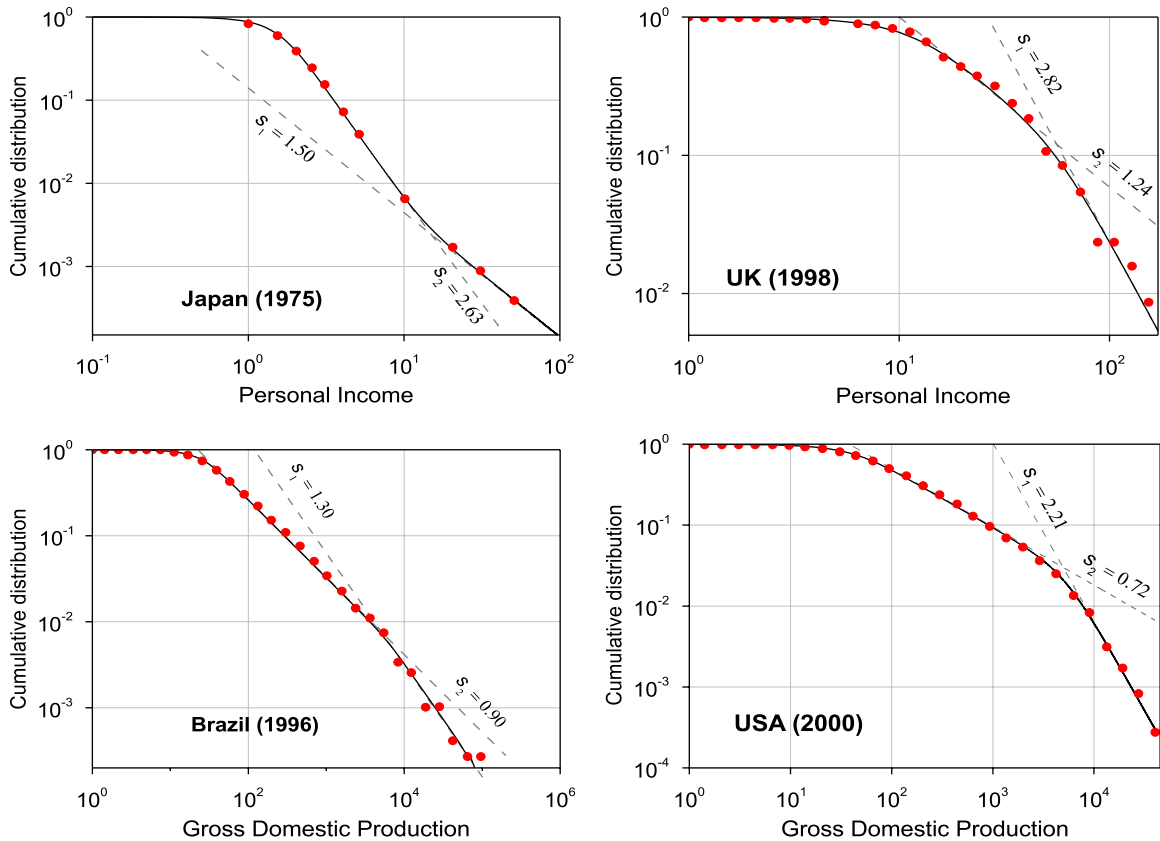


Fig. 2. Log–log plot of personal income distribution for Japan (1975) [18] and USA (2000) [3] and gross domestic production distribution for Brazil (1996) and Germany (1998) [3]. The solid line represents the fit obtained with the two-power law function (3.8). The straight dashed lines are plotted for convenience to indicate the asymptotic power-law prolongation.

Table 1  
Parameters for the cumulative distribution  $P(x)$

Country	$\kappa_1$	$\kappa_2$	$a_1$	$\mu$	$\beta$
Japan (1975)	1.14	2.00	390	3.00	$3.5 \times 10^{-4}$
UK (1998)	1.70	0.75	8	2.12	$2.5 \times 10^{-4}$
Brazil (1996)	2.20	1.53	$2 \times 10^4$	1.99	$2.3 \times 10^{-8}$
USA (2000)	2.00	0.65	231	1.44	$6.0 \times 10^{-6}$

### 5. Generalization

Let us briefly discuss the generalization of the method introduced in Section 3 in order to generate functions with more than two-power law behavior. This can be accomplished starting from the building block function

$$\Pi_{\{\sigma_i\}}(x) = \exp_{\{\kappa_i, r_i\}}(a_i \ln_{\{\kappa_i, r_i\}}(x)), \tag{5.1}$$

and introducing the quantity

$$\Pi_{\{\vec{\sigma}\}}(x) = \Pi_{\{\sigma_1\}} \circ \Pi_{\{\sigma_2\}} \circ \dots \circ \Pi_{\{\sigma_{n-1}\}}(x), \tag{5.2}$$

where  $\vec{\sigma} \equiv (\sigma_1, \sigma_2, \dots, \sigma_{n-1})$  is a  $(n-1)$ -vector whose  $i$ th entry  $\sigma_i = (\kappa_i, r_i, a_i)$  contains the relevant informations about the slope and the width of the  $i$ th power law region. It is easy to verify that

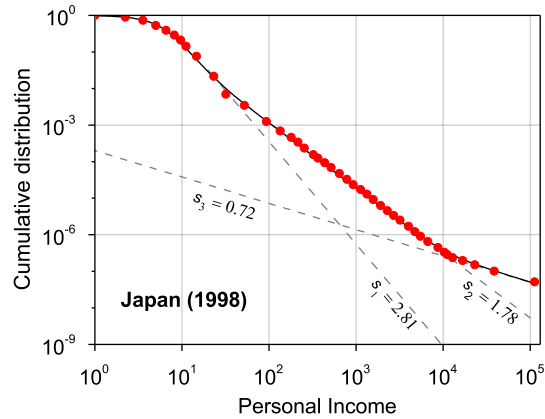


Fig. 3. Log–log plot of the cumulative distribution of Japan for the year 1998. It is observed a deviation from the Pareto behavior in the highest income region ( $x > 10^4$ ).

the function

$$f(x) = \Pi_{\{\bar{\sigma}\}} \circ \exp_{\{\kappa_n, r_n\}}(-x), \quad (5.3)$$

exhibits a  $n$ -power law behavior.

In Fig. 3, we report the fit of the 1998 Japanese income data obtained in Ref. [19] by employing the function

$$P(x) = \Pi_{\{\bar{\sigma}\}} \circ \exp_{\{\kappa_3\}}(-\beta x^\mu), \quad (5.4)$$

derived from Eq. (5.3) for  $n = 3$  and  $r_i = 0$ . The fitting data are  $\kappa_1 = 0.71$ ,  $\kappa_2 = 1.12$ ,  $\kappa_3 = 2.77$ ,  $a_1 = 10$ ,  $a_2 = 4 \times 10^5$ ,  $\beta = 6.00 \times 10^{-9}$  and  $\mu = 2.00$ .

## 6. Conclusions

We have derived a simple method which permits to generate functions with a multi-power law behavior starting the deformed logarithm  $\ln_{\{\kappa, r\}}(x)$  and the deformed exponential  $\exp_{\{\kappa, r\}}(x)$ , recently derived in Ref. [11], which exhibit a single power law profile. An explicit two-power law function has been constructed starting from the  $\kappa$ -exponential and its inverse, the  $\kappa$ -logarithm. We have employed this function to fit the inverse cumulative distribution of the personal income and of the gross domestic production of several countries, showing a good agreement among the analytical and the empirical data for a wide range of values.

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