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Physica A 382 (2007) 347–353
PHYSICA A
www.elsevier.com/locate/physa

The matrix rate of return

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Available online 1 March 2007

Abstract

In this paper we give definitions of matrix rates of return which do not depend on the choice of basis describing baskets. We give their economic interpretation. The matrix rate of return describes baskets of arbitrary type and extends portfolio analysis to the complex variable domain. This allows us for simultaneous analysis of evolution of baskets parameterized by complex variables in both continuous and discrete time models.

Keywords: Interest rates; Finance; Capital processes; Investment techniques; Harmonic oscillator

1. Introduction

The goal of capital investment is maximization of profit and minimization of possible losses. This goal cannot be achieved by investing the whole capital in the most profitable enterprises. Such situations do not happen. The future profit of a market investment is uncertain, therefore, the investor creates composite baskets consisting of capital investments of possibly most diversified character. This kind of procedure diversifies the risk of enterprise. The description of the evolution of multidimensional capital of this kind is essential to quantitative analysis of the correlations related with investment processes, in particular, for which the tools of traditional financial mathematics are unapplicable. Depending on the point of view, in every capital basket we can, besides of quantitative changes of individual components, observe the flows between its components. The flows of capital can be recorded even if no decisions about capital operations are taken. Such situations require matrix description and enforce the generalizations of the calculus of interest rates in such a way that they are sensitive to both quantitative changes of individual elements and flows between them. This situation leads inevitably to the matrix generalization of the interest rate calculus, see Ref. [1].

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0378-4371/$ - see front matter
doi:10.1016/j.physa.2007.02.028

2. Linear homogeneous capital process

Let us consider a capital of a banker which is a mixture of two elements: k_1—an amount lent to client, k_2—the remaining of assets. Let the variable $l \in \mathbb{Z}$ denotes arbitrary time interval. The process of change of the banker's capital related with the amounts k_1 and k_2 can be described as follows:

$k_1(l)$: According to the rate of interest $\alpha_1(l)$ at which the banker gave credit, the growth of the component k_1 is equal to $\alpha_1(l)k_1(l)$. In addition, the amount of unreturned part of credit decreases by the value of repayment $\beta(l)k_1(l)$ which is determined by the rate $\beta(l)$.

$k_2(l)$: The capital k_2 increases by the amount of repayment of credit $\beta(l)k_1(l)$. Being in addition placed in, for example liquid stocks of annual rate of return $\alpha_2(l)$, it grows by the amount $\alpha_2(l)k_2(l)$.

The components of the banker's capital form a basket $k(l) = (k_1(l), k_2(l))$. It is represented by an element of the two-dimensional real vector space $\mathbb{R}^2$. The evolution of this basket during the time $l \in \mathbb{Z}$ can be described by the system of equations

$$\begin{cases} k_1 1(l+1) = (1+\alpha_1(l) - \beta(l))\; k_1(l), \\ k_2(l+1) = \beta(l)\; k_1(l) + (1+\alpha_2(l))\; k_2(l). \end{cases} \tag{1}$$

We interpret the negative values of components of the basket k_m, $m = 1, 2, \ldots$ as debts of the banker.

The nonlinear (with respect to the remaining debt) repayment rates, that is for example the costs of service of credit or taxes, can be presented in a form of linear repayment, after appropriate modification of the factor $\beta(l)$. In particular, various borrower's obligations to the banker can be taken into account with the help of the expression in variable proportion to the amount of paid-off credit. This enables us to apply our formalism in much wider context. This evolution of the capital can be described in such a way that any changes of components of the basket are expressed as percent changes of those components. In the vector space of baskets we can choose a new basis such that we do not observe any flows of capital, but autonomous growth of individual components only. Let us assume that $\alpha_1(l) \neq \alpha_2(l)$ and $k_1'(l)$ be as in previous basis, the amount of the capital lent to client $k_1'(l) = k_1(l)$. $k_2'(l)$ is the sum of $\beta(l)$ part of a loan and $\alpha_2(l) - \alpha_1(l) + \beta(l)$ part of remaining banker's assets. That is, $k_2'(l) = \beta(l)k_1(l) + (\alpha_2(l) - \alpha_1(l) + \beta(l))k_2(l)$. In these new variables equations (1) describing the changes in the basket separate and take the following form:

$$\begin{cases} k_1'(l+1) = (1+\alpha_1(l) - \beta(l))\; k_1'(l), \\ k_2'(l+1) = (1+\alpha_2(l))\; k_2'(l). \end{cases}$$

In this way we do not observe any flows of capital and the components k_1' and k_2' of the basket grow according to the rates of interest $\alpha_1(l) - \beta(l)$ and $\alpha_2(l)$ for k_1' and k_2', respectively. The formalism of financial mathematics should not depend on choice of basis to describe the baskets. It seems that the matrix rates presented below are necessary to obtain the basis independent description of the economical reality.

3. The matrix rate of return

Capital processes are described by linear homogeneous difference equations. In the matrix notation they take the form

$$\mathbf{k}(l+1) - \mathbf{k}(l) = \underline{\mathbf{R}}(l)\,\mathbf{k}(l) \quad \text{hence} \quad \mathbf{k}(l+1) = (\mathbf{I} + \underline{\mathbf{R}}(l))\;\mathbf{k}(l), \tag{2}$$

where $\mathbf{k}(l) \in \mathbb{R}^M$, $\underline{\mathbf{R}}(l)$ and $\mathbf{I}$ are real matrices of dimension $M \times M$. $\mathbf{I}$ denotes the unit matrix. Matrix $\underline{\mathbf{R}}(l)$ is called the matrix credit rate of return or the matrix lower rate. The adjective lower refers to the way of the description of the growth of the vector $\mathbf{k}(l)$. These changes are expressed as the effects of linear transformations applied to the vector $\mathbf{k}(l)$ of the basket at the moment l prior to this change. The space $\mathbb{R}^M$ of all possible baskets is called the phase space of baskets, and linear homogeneous difference or differential systems of equations of first order are called equations of motion of the basket. For $M = 2$ the matrix $\underline{\mathbf{R}}(l)$

generating the evolution (1) equals to

$$\underline{\mathbf{R}}(l) = \begin{pmatrix} \alpha_1(l) - \beta(l) & 0 \\ \beta(l) & \alpha_2(l) \end{pmatrix}. \tag{3}$$

If the state of the basket in the initial moment p equals to $\mathbf{k}(p)$, then the solution of the evolution equation (2) takes the form

$$\mathbf{k}(r) = \left(\mathscr{T} \prod_{s=p}^{r-1} (\mathbf{I} + \underline{\mathbf{R}}(s)) \right) \mathbf{k}(p), \tag{4}$$

where $\mathscr{T}$ denotes the chronological ordering operator. It orders all matrices chronologically: the matrices with later arguments are moved to the left of the matrices with earlier arguments, that is

$$\mathscr{T} \prod_{s=p}^{p+n} \mathbf{A}(s) = \mathbf{A}(p+n)\mathbf{A}(p+(n-1)), \ldots, \mathbf{A}(p+1)\mathbf{A}(p),$$

where $\mathbf{A}(s)$ is an arbitrary sequence of matrices.

The chronological product $\mathscr{T}\prod_{s=p}^{r-1}(\mathbf{I} + \underline{\mathbf{R}}(s))$ contains more detailed information about changes of capital in the basket than the usually used quotients $\sum_{m=1}^{M} k_m(r) / \sum_{m=1}^{M} k_m(p)$. The possibility to use the standard calculus of interest rates depends on way of description of the basket, that is on choice of a basis in our language. For this reason, in order to obtain an invariant description, the rate of growth $\underline{\mathbf{R}}(l)$ is promoted to a composite object with definite transformation rules corresponding to the changes of reference frame (an observer).

Let us consider processes for which the matrix $\mathbf{I} + \underline{\mathbf{R}}(l)$ is nonsingular. Then, introducing the concept of the matrix discount rate of return or the matrix upper rate $\overline{\mathbf{R}}(l)$, it is possible to rewrite the equation of motion (2) in the following form:

$$\mathbf{k}(l+1) - \mathbf{k}(l) = \overline{\mathbf{R}}(l)\ \mathbf{k}(l+1). \tag{5}$$

Comparing formulae (2) and (5) we obtain the relation between both types of matrix rates introduced above

$$(\mathbf{I} + \underline{\mathbf{R}}(l))(\mathbf{I} - \overline{\mathbf{R}}(l)) = (\mathbf{I} - \overline{\mathbf{R}}(l))(\mathbf{I} + \underline{\mathbf{R}}(l)) = \mathbf{I}. \tag{6}$$

Solving the above equation with respect to $\underline{\mathbf{R}}(l)$ we obtain

$$\underline{\mathbf{R}}(l) = \overline{\mathbf{R}}(l) + \overline{\mathbf{R}}^2(l) + \overline{\mathbf{R}}^3(l) + \cdots. \tag{7}$$

From the point of view of capitalization from upper, we can interpret the above formulae as the contribution of discount rate into increase of capital by summation of all interests from interests (the geometrical sequence). According to Eq. (7) the formulae for $\overline{\mathbf{R}}(l)$ take the form $-\overline{\mathbf{R}}(l) = (-\underline{\mathbf{R}}(l)) + (-\underline{\mathbf{R}}(l))^2 + (-\underline{\mathbf{R}}(l))^3 + \cdots$. Notice that for fixed argument l the matrix rates are commuting.

When we perform the formal change of the direction of time, the credit and discount rate change their signs and they change their roles. Therefore, the formulae containing these matrices are symmetric with respect to the time reflection. The appropriate formulae for (4) can be obtained with the help of (6) as the solution of the equation of motion

$$\mathbf{k}(p) = \left(\mathscr{T}' \prod_{s=p}^{r-1} (\mathbf{I} - \overline{\mathbf{R}}(s)) \right) \mathbf{k}(r), \tag{8}$$

where $\mathscr{T}'$ is the antichronological operator which orders matrix rates in the direction opposite to that corresponding to the operator $\mathscr{T}$.

If the matrix rates at different moments of time commute, that is in the case of $M = 1$ or for any M if matrix rates are time independent, we can neglect the operators $\mathscr{T}$ and $\mathscr{T}'$ in the solutions (4) and (8).

4. Interpretation of the matrix rate of return

The matrix rate of return $\underline{\mathbf{R}}(l) = (\underline{\mathbf{R}}_{mn}(l))$ can be given as sum of two matrices $\underline{\mathbf{R}}(l) = \underline{\mathbf{C}}(l) + \underline{\mathbf{D}}(l)$, with $\underline{\mathbf{C}}(l)$ the matrix of flows (the name is justified by the property that sum of the elements of each column is equal to zero) and $\underline{\mathbf{D}}(l)$ the diagonal matrix called the matrix of growths. This decomposition is unique, when the basis is fixed. Introduction of the matrix rate of return is essential, when we cannot transform the matrix $\underline{\mathbf{C}}(l)$ to matrix zero. Moreover, the off-diagonal elements of the matrix $\underline{\mathbf{R}}(l)$, $\underline{\mathbf{R}}_{mn}(l)$ for $m \neq n$, see Eq. (3), describe which part of capital of the nth component of the basket flows to the mth component of the basket. Diagonal elements $\underline{\mathbf{R}}_{mm}(l)$ describe the growths. The rate of growth of the mth component of the basket equals $\alpha_m(l) = \underline{\mathbf{R}}_{mm}(l) + \sum_{n \neq m} \underline{\mathbf{R}}_{nm}(l)$, that is, the diagonal element corrected by all outflows of the capitals related to the component $\mathbf{k}_m(l)$.

4.1. Example

(i) Let us restrict the process from Section 2 to the cases when $\beta(l) = 0$ and the coefficients α_i do not depend on time l and the second coefficient is two times bigger than the first one. Then the matrix lower rate is equal to

$$\underline{\mathbf{R}} = \begin{pmatrix} \alpha & 0 \\ 0 & 2\alpha \end{pmatrix}.$$

According to the principal formula (6) the corresponding matrix upper rate is given by

$$\overline{\mathbf{R}} = \begin{pmatrix} \frac{\alpha}{1+\alpha} & 0 \\ 0 & \frac{2\alpha}{1+2\alpha} \end{pmatrix}.$$

The process contains the matrix of growths only because the matrix of flows is zero. It is possible to analyze the profits on the grounds of autonomic evolution in one-dimensional subspaces of the phase space—the classical concept of interest rates is applicable here.

(ii) We can look at this process in a different way. Namely, by describing it in the coordinates in the new basis in the space of baskets. Let the reference basis consist of client's debts and all banker's capital assets. Then the equation of motion of the basket (1) takes the form

$$\begin{cases} \widetilde{\mathbf{k}}_1(l+1) = (1+\alpha)\, \widetilde{\mathbf{k}}_1(l), \\ \widetilde{\mathbf{k}}_2(l+1) = -\alpha\, \widetilde{\mathbf{k}}_1(l) + (1+2\alpha)\, \widetilde{\mathbf{k}}_2(l). \end{cases} \tag{9}$$

The matrix of flows determined by the above equation is nonzero now. The matrix of flows and the matrix of growths equal to

$$\underline{\mathbf{C}} = \begin{pmatrix} \alpha & 0 \\ -\alpha & 0 \end{pmatrix}, \quad \underline{\mathbf{D}} = \begin{pmatrix} 0 & 0 \\ 0 & 2\alpha \end{pmatrix},$$

respectively. The debt of a client $\tilde{k}_1$ changes in a similar way to the previous example, though not as result of an autonomic growth, but due to an outflow of the banker's capital. Banker's capital grows according to the same rate as investment $k_2(l)$.

(iii) On the basis given by Eq. (6) the matrix upper rate for the equation of motion (9) takes the form

$$\overline{\mathbf{R}} = \begin{pmatrix} \frac{\alpha}{1+\alpha} & 0 \\ -\frac{\alpha}{(1+\alpha)(1+2\alpha)} & \frac{2\alpha}{1+2\alpha} \end{pmatrix}$$

and it is the sum of the following matrices of flows and growths:

$$\overline{\mathbf{C}} = \begin{pmatrix} \frac{\alpha}{(1+\alpha)(1+2\alpha)} & 0 \\ -\frac{\alpha}{(1+\alpha)(1+2\alpha)} & 0 \end{pmatrix}, \quad \overline{\mathbf{D}} = \begin{pmatrix} \frac{2\alpha^2}{(1+\alpha)(1+2\alpha)} & 0 \\ 0 & \frac{2\alpha}{1+2\alpha} \end{pmatrix}.$$

Different points of view at the same capital process presented in (i)–(iii) are equally correct and sensible. Comparing the credit convention in variant (ii) with the discount convention (iii) we can note the essential difference between the matrices $\underline{\mathbf{D}}$ and $\overline{\mathbf{D}}$. Growth of first component is only the effect of flows in the first case while in the second case this component has partial autonomy in its growth. The indicated difference in interpretation can be a reason of many financial embezzlement, exactly in the same way as it happens in case of nonpayment of interests of overdue interests in simple capitalization. We will refer the asymmetry of this kind in the description of flows as the paradox of difference rates.

The interpretation of the matrix rate $\overline{\mathbf{R}}$ is analogous to that of the matrix $\underline{\mathbf{R}}$ with the only difference that the rate $\overline{\mathbf{R}}$ defines the capital changes with respect to the oncoming moment of the settlement of accounts. The convention of discount replaces the convention of credit. The free choice of convention is not possible for nonreversible processes, i.e. when the matrices $\underline{\mathbf{R}}$ and $\overline{\mathbf{R}}$ are singular.

5. The formalism of continuous description of credit

For the capital calculus to be transparent and readable for practitioners the formal solution of the equation of motion (2) can be modeled numerically or presented in dense form with the help of the limiting process which transforms the discrete models described by linear difference equations into continuous ones with differential equations. Assume that we consider time scales such that the period of time between the changes of components of the basket is infinitesimal and equals to $\tau = t_{l+1} - t_l$. After rescaling of the time domain of the basket the equation of motion (2) takes the form

$$\frac{\mathbf{k}(t_l+\tau)-\mathbf{k}(t_l)}{\tau} = \frac{\underline{\mathbf{R}}(l)}{\tau}\ \mathbf{k}(t_l).$$

In the limit $\tau \to 0$ we obtain

$$\frac{\mathrm{d}\mathbf{k}(t)}{\mathrm{d}t} = \mathbf{R}(t)\ \mathbf{k}(t) \quad \text{where} \quad \mathbf{R}(t) := \lim_{\tau\to 0} \frac{\underline{\mathbf{R}}(l)}{\tau}\bigg|_{t=t_l}. \tag{10}$$

Matrix $\mathbf{R}(t)$ is called the differential matrix rate of return.

The formal solution of Eq. (10) is given by the formula

$$\mathbf{k}(t) = (\mathscr{T}\,\mathrm{e}^{\int_{t_0}^{t}\mathbf{R}(t')\,\mathrm{d}t'})\mathbf{k}(t_0). \tag{11}$$

The chronologically ordered exponential function is infinite series in the differential matrix rate

$$\mathscr{T}\mathrm{e}^{\int_{t_0}^{t}\mathbf{R}(t')\,\mathrm{d}t'} = \mathbf{I} + \int_{t_0}^{t}\mathbf{R}(t_1)\,\mathrm{d}t_1 + \int_{t_0}^{t}\mathbf{R}(t_1)\int_{t_0}^{t_1}\mathbf{R}(t_2)\,\mathrm{d}t_2\mathrm{d}t_1 + \cdots.$$

If the differential matrix rate is constant, the chronological operator $\mathscr{T}$ is the identity. The expression on the right-hand side of Eq. (11) which describes the time evolution of the basket becomes transformed to the standard matrix exponential function $\mathscr{T}\,\mathrm{e}^{\int_{t_0}^{t}\mathbf{R}(t')\,\mathrm{d}t'} = \mathrm{e}^{(t-t_0)\mathbf{R}_0}$.

6. Complex rate of return

Let us consider baskets which have oscillating components. Part of their capital becomes an unwanted at some moment and a desirable good at another time. This phenomenon is called pumping of capital in the language of financial market. Evolution of basket of this kind is described by one of the most popular models of physics—the harmonic oscillator. Let us consider the differential matrix rate

$$\mathbf{R}(t) = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} + \begin{pmatrix} -b & -b \\ b & b \end{pmatrix}.$$

It is easy to imagine the corresponding bank contracts leading to the flows of capital in basket and autonomic increases of components of the basket generated by these flows. Let us consider the complex extension $\mathbb{C}^2$ of the phase space $\mathbb{R}^2$. Then $\widetilde{\mathbf{k}}_1 = (1, \mathrm{i})$ and $\widetilde{\mathbf{k}}_2 = (1, -\mathrm{i}) = \widetilde{\mathbf{k}}_1^*$ are the eigenvectors of the matrix rate $\mathbf{R}(t)$ with eigenvalues $-\mathrm{i}\, b$ and $\mathrm{i}\, b$, respectively. The description of process is simplified because the basket splits into two independent components with abstract complex capital. Although that baskets of the complex capital are the abstract concepts, they have the interpretation in the real basis due to matrix of transition. The equation of motion of the basket in the basis of eigenvectors $\{\widetilde{\mathbf{k}}_1, \widetilde{\mathbf{k}}_2\}$ takes the form

$$\widetilde{\mathbf{k}}_1(t) = \mathrm{e}^{-\mathrm{i}b(t-t_0)}\, \widetilde{\mathbf{k}}_1(t_0), \quad \widetilde{\mathbf{k}}_2(t) = \mathrm{e}^{\mathrm{i}b(t-t_0)}\, \widetilde{\mathbf{k}}_2(t_0).$$

In the initial, real basis one has

$$\mathbf{k}(t) = \begin{pmatrix} \cos(b(t-t_0)) & -\sin(b(t-t_0)) \\ \sin(b(t-t_0)) & \cos(b(t-t_0)) \end{pmatrix} \mathbf{k}(t_0).$$

This equation describes motion along the circle centered at the beginning of the Cartesian coordinates of the basket. The period of return to the same point of the phase space equals to $T = 2\pi/|b|$.

7. The indefinite matrix rate

We define the indefinite logarithm, see Ref. [2], to be a mathematical object representing the abstract concept of the logarithm with an unfixed base. For any given real number $x > 0$, the indefinite logarithm of x written as $[\log\, x]$, is a special type of mathematical object called a logarithmic quantity object, which we define as follows [2]:

$$[\log\, x] := \{(b, y) | b > 0, y = \log_b x\}.$$

Indefinite logarithmic quantities are inherently scale-free objects, that is, they are nonscalar quantities and they can serve as a basis for logarithmic spaces, which are natural systems of logarithmic units suitable for measuring any quantity defined on a logarithmic scale. Although the above definition is restricted to positive real numbers, it could be extended to nonzero complex numbers too. The concept of the rate of interest is connected with the time-scale. To get rid of explicit time-scale dependence we introduce the indefinite matrix rate, which is the generalization of the indefinite logarithm to the multidimensional case

$$[\log(\mathscr{T} \mathrm{e}^{\int_{t_0}^{t} \mathbf{R}(t')dt'})].$$

In the above definition the argument of the logarithm is a matrix.

8. Conclusions

Every matrix can be deformed to a diagonalizable complex matrix by arbitrarily small deformations of their elements [3,4], that is the set of diagonalizable linear transformations of the complexified phase space $\mathbb{C}^M$ is dense in the space of all linear maps of $\mathbb{C}^M$. From the above it follows that, the evolution of every capital basket can be represented as set of noninteracting complex capital investments. Therefore, in the complex extended phase space the decomposition of the matrix rate $\underline{\mathbf{R}}(t) = \underline{\mathbf{C}}(t) + \underline{\mathbf{D}}(t)$ can always be done in such a way that the matrix of flows is zero. As in traditional financial mathematics the real parts of nonzero elements of a diagonal matrix rate measure the loss or gain of complex investments. The imaginary parts inform about periodicity of changes in proportion between real component of complex investments and its imaginary partner. To every periodic changes of proportion of the elements of the complex components of capital there will be a corresponding complex conjugated partner. The evolution of real capital of basket is most easily observed in terms of its components with respect to the basis of eigenvectors of matrix rates.

By the choice of appropriate moments of entering and exit from capital process the oscillations like above can be used as a particularly effective mechanism of enlarging of the capital giving similar results as financial leverage.

Acknowledgment

We are greatly indebted to Prof. Zbigniew Hasiewicz for helpful remarks.

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