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Fractal Model for Snow

A. Carbone[1,a], B. Chiaia[2,b], B. Frigo[2,c] and C. Türk[1,d]

[1]Politecnico di Torino, Department of Physics, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

[2]Politecnico di Torino, Department of Structural and Geotechnical Engineering, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

[a]anna.carbone@polito.it, [b]bernardino.chiaia@polito.it,

[c]barbara.frigo@polito.it, [d]christian.turk@polito.it

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Abstract. We analyze the distribution of grains in solid cubes of ice in terms of deterministic and stochastic 3d fractal models. We argue that the fractal dimension D or the Hurst exponent H optimally describe the void distribution in the snow sample and can be used as a parameter to describe the mechanical properties of snow at different scales.

Introduction

Snow is described as a porous medium consisting of air and three water phases (ice, vapour or liquid), while the ice phase consists of an assemblage of grains (the ice matrix), initially arranged on a random load bearing skeleton. For a proper description of snow behavior, it would be necessary to know all its physical-mechanical properties. Unfortunately these properties are very difficult to measure in situ and in the laboratory. Therefore, the density is practically the only parameter used to determine the physical properties of snow in situ. However, it is not sufficient to define the mechanical characteristics of different snow microstructures, that can yield the same global density [1, 2].

Fractal geometry has already been used to model snow crystals and for the construction of geometrical topography of snowflakes (von Koch snowflake curves). Thanks to 3d radiographies of cubic samples, with different sizes and densities, the fractal character of snow has been recently studied by the Centre Etudes de Neige of Météo France (C.E.N.) [3]. These images show the granular structure of snow and the spatial distribution of its voids. By means of the box-counting method [4, 5], the fractal dimension D of four snow samples characterized by grains with different diameters could be determined. The fractal dimension D was used to describe the distribution of the voids in each sample with good accuracy.

The density of snow can be simulated by means of a deterministic sponge model. Here, we introduce a stochastic model that can capture the randomness of the local fractal structure of snow samples. In particular, we have developed two 3d fractal generators: the pseudo random fractal algorithm and the random midpoint displacement algorithm, to estimate snow density.

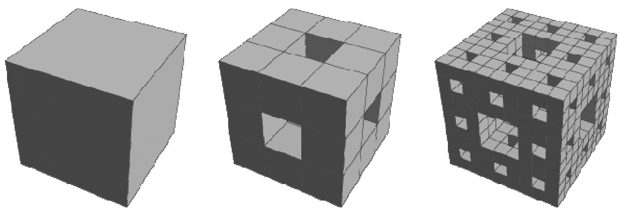
Deterministic fractal model

The intergranular porosity of snow exhibits fractal behaviour. The fractal model, known as the Menger sponge, can be used to characterise the scale invariant porosity, which is ultimately related to the density of snow. Fig. 1 illustrates the basic generator. A bulky cube is divided into $3 \times 3 \times 3 = 27$ equal subcubes. Then, one removes each subcube at the middle of every face plus the one in the centre of the cube. As a result, one gets the 2nd order deterministic Menger sponge. In the general process, this single step will be repeatedly applied, creating at each iteration i, a Menger sponge of level i.

Let us consider a solid cube with a given initial density ρ_0 and linear size r_0. The void index or porosity ϕ_i, is defined as the relative volume of voids per unit volume, at a given iteration i. ϕ_i, is related to ρ_0 and ρ_i (density at each i) according to:

$$\phi_i = 1 - \left(\frac{\rho_i}{\rho_0}\right). \qquad (1)$$

By generating the 1st level sponge, characterized by linear size $r_1 = r_0/3$, the void index $\phi_1 = 7/27$ and the density $\rho_1 = 20\rho_0/27$ (Fig. 1.b). Then we can construct the 2nd level Menger sponge (Fig. 1.c). This iteration shows the fractal cube as $r_2 = r_0/9$, with porosity $\phi_2 = 329/729$ and density $\rho_2 = 400\rho_0/729$.



(a) (b) (c)

Fig. 1. Deterministic Menger sponge after two iterations: (a) r_0 , ρ_0 and $\phi_0 = 0$; (b) $r_1 = r_0/3$, $\rho_1 = 20\rho_0/27$ and $\phi_1 = 7/27$; (c) $r_2 = r_0/9$, $\rho_2 = 400\rho_0/729$ and $\phi_2 = 329/729$.

This procedure can be generalized by removal of an arbitrary number N_e of subcubes. In this case, the basic generator starts from a bulky cube, which is then divided into $3 \times 3 \times 3$ subcubes, where N_f out of them are filled, while $N_e = (3^3 - N_f)$ are empty. $r_i = r_0/3^i$ is the linear dimension of the sample considered. Therefore, the density and void index of the i^{th} level Menger sponge are defined respectively as:

$$\frac{\rho_i}{\rho_0} = \left(\frac{N_f}{3^3}\right)^i \quad \text{and} \quad \phi_i = 1 - \left(\frac{N_f}{3^3}\right)^i, \qquad (2)$$

or, as a function of the linear dimension of cube r_0 and subcubes r_i:

$$\left(\frac{\rho_i}{\rho_0}\right) = 1 - \phi_i = \left(\frac{r_i}{r_0}\right)^{3-\ln N_f/\ln 3} = \left(\frac{r_i}{r_0}\right)^{3-D} = \left(\frac{r_i}{r_0}\right)^{H} \qquad (3)$$

Note that Eq. 3 is a fractal relation, with the fractal dimension $D = \ln N_f / \ln 3$ appearing in the exponent. D is related to the Hurst exponent by $H = 3 - D$. To model snow samples, we consider a solid ice cube characterized by density, $\rho_{ice} = \rho_0 = 917$ kg/m^3, and linear size $r_{ice} = r_0$. We can simulate the snow as a fractal form of ice [1] characterized by density $\rho_{snow} = \rho_i$, and linear size $r_{snow} = r_i$ at the i^{th} scale. In the case of snow, we can thus rewrite Eq. 3 as follows:

$$\left(\frac{\rho_{snow}}{\rho_{ice}}\right) = 1 - \phi_{snow} = \left(\frac{r_{snow}}{r_{ice}}\right)^{3-\ln N_f/\ln 3} = \left(\frac{r_{snow}}{r_{ice}}\right)^{3-D} = \left(\frac{r_{snow}}{r_{ice}}\right)^{H}. \qquad (4)$$

The fractal dimension D (or the Hurst exponent H) is the scale-independent parameter characterizing the topology of the solid microstructure and enables a measure of ice lacunarity in the snow sample. The fractal character of snow has recently been studied by C.E.N. [3], using images obtained from the 3d radiographies on various snow samples, cubic samples with size 2.5 mm and different densities. These images show the granular structure of snow and the spatial distribution of its voids. By means of the box-counting method, the fractal dimension of four snow samples, characterized by grains with different diameter, could be determined [4].

By applying this model and using the same size of the samples: r_{snow} = 2.5 mm and $r_{ice} = r_0$ = 1 m, we can calculate the fractal dimension D of different classes of snow, see Table 1. Hence, D = 2.727 corresponds to dry snow, with density equal to 178,65 kg/m^3, while D = 2.893 corresponds to snow with density equal to 483,53 kg/m^3. By analyzing the density and void index as a function of the linear sample dimension r_i, we observe that as the sample size increases, snow differs more and more from ice. At small scales, ice and snow approximately show the same behaviour while the spatial variability of the density does not greatly influence the mechanical properties. Therefore, we argue that snow density is a function of the scale and the probability to find large defects (e.g. super-weak zones in a weak layer) increases with the dimension of the snow cover, as for example providing more intrinsic brittleness for large snow slopes [6, 7]. The numerical results reported in Table 1 confirm that D is a measure of the distribution of the ice mass into the snow sample. We also observe that the values of the fractal dimension measured by the box-counting method [4] are consistent with the values calculated by the deterministic sponge model, see Table 2.

Table 1. Classification of snow and ice according to the density with D and H for a sample with linear size r_{snow} = 2.5 mm [4, 8].

Snow type	Density [kg/m^3]	Fractal dimension	Hurst exponent
Dry snow	$50 < \rho < 200$	$2.5145 < D < 2.7458$	$0.4855 < H < 0.2542$
Snow	$200 < \rho < 550$	$2.7458 < D < 2.9147$	$0.2542 < H < 0.0853$
Firn	$550 < \rho < 820$	$2.9147 < D < 2.9813$	$0.0853 < H < 0.0187$
Porous ice	$820 < \rho < 917$	$2.9813 < D < 3.0000$	$0.0186 < H < 0.0000$
Ice	917	3	0

Table 2. The fractal dimension of different snow samples: comparison between box counting method [4] and the deterministic sponge model.

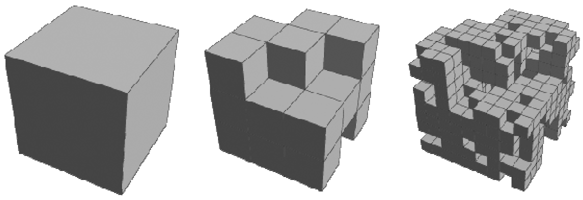
Snow samples	Density [kg/m^3]	Box counting		Deterministic sponge	
		D	H	D	H
Fine grains	200	2.62	0.38	2.7458	0.2542
Fine grains (Huez)	200	2.83	0.17	2.7458	0.2542
Fine grains	300	2.80	0.20	2.8135	0.1865
Rounded grains	300	3.00	0.00	2.8135	0.1865

Random fractal model

Stochastic fractals, generated from the randomization of deterministic fractals, have the advantage to be closer to real materials, as they are partially disordered. We have seen that the deterministic Menger sponge model, characterised by a scale invariant porosity, could be used to define the density of different snow samples (both the type and microstructure). Still, this model does not capture the randomness of the local fractal structure of real snow samples, since it has a limited fractal dimension. Therefore, we propose a method, the pseudo random fractal algorithm, to construct a snow sponge from a solid cube by an iterative process of cube removal and rescaling. The porosity of a "true" fractal should be free of scale. This is obtained by iterating the generating process an infinite number of times. This property is not fulfilled for such a cube removal process. By introducing a lower cut-off size meaning that the iteration process is iterated only i times, we can define such a structure as a "pseudo fractal" [9].

On the account of these limitations, we therefore present a new more rigorous method, which is a generalization to 3d of the random midpoint displacement algorithm. This recursive generating technique provides a porosity which is exactly scale invariant [10].

Pseudo random 3d fractals. This method is an extension of the deterministic Menger sponge model where the generator is modified by randomly removing a given number of subcubes at each iteration step. As an example, at the first iteration step, the cube is divided into 3 × 3 × 3 subcubes. We then randomly remove a fixed number of subcubes $N_e = (3^3 - N_f)$. We can take, for example $(3^3 - N_f) = 7$ as shown in Fig 2. At the next level, this procedure is repeated on the $N_f = 20$ remaining subcubes. However, each of the 20 remaining subcubes undergoes a randomly distributed removal process. The process is iterated only i times, corresponding to the cut-off size, so that we can achieve a realistic porosity. If we would continue with an infinite number of iterations, and removing a large number of subcubes $(3^3 - N_f)$ at each step, the sponge would at some point vanish (there are no fixed boundaries). The random based evolution for the first two iterations is shown in Fig. 2. We see that the local structure changes while preserving the fractal dimension $D = 2.727$. As we change the fixed number of subcubes to be removed, one can not only construct various types of sponges, but also get different fractal dimensions.



(a) (b) (c)

Fig. 2. Random sponge generator obtained by randomly removing 7 subcubes at each iteration step: (a) r_0 , ρ_0 and $\phi_0 = 0$; (b) $r_1 = r_0/3$, $\rho_1 = 20\rho_0/27$ and $\phi_1 = 7/27$; (c) $r_2 = r_0/9$, $\rho_2 = 400\rho_0/729$ and $\phi_2 = 329/729$. The fractal dimension D is 2.727.

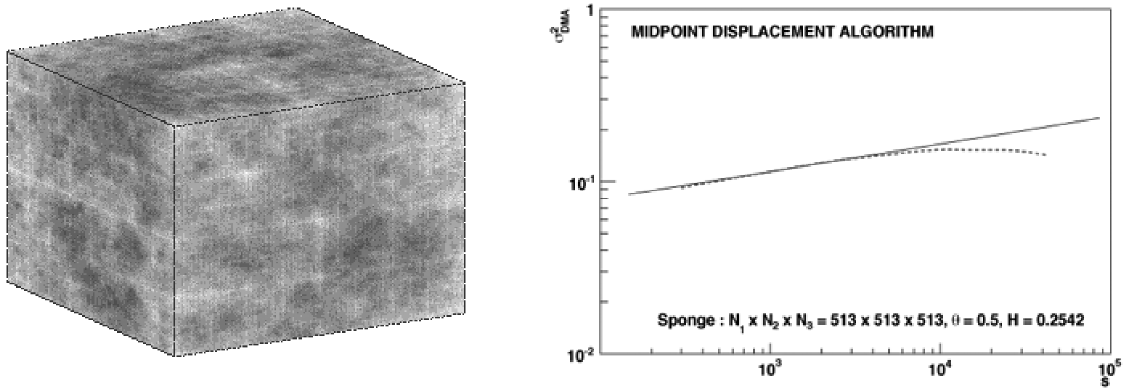
3d random midpoint displacement algorithm. The random midpoint displacement algorithm (RMDA) is a recursive generating technique and is a natural extension of the von Kock structure. The execution speed of the RMDA and its ability to add "details" to an euclidean regular shape makes it suitable to model various fractals. Here, we propose the first 3d implementation of the RMDA. The 3d fractal obtained by the proposed method holds the property to be scale invariant. According to the present method, one defines the midpoints at level l of n points as:

$$p_{mid}^{(l,n)} = \frac{1}{n}(p_1 + p_2 + ... + p_n) + \Delta_l , \tag{5}$$

where Δ_l is a Gaussian random variable with zero mean and variance level l:

$$\Delta_l^2 = \frac{\sigma^2}{(2^l)^{2H}}(1 - 2^{2H-2}). \tag{6}$$

The Hurst exponent H is the input for the generator, which determines the stochastic variability of the obtained microstructure and therefore, the granularity of the resulting fractal. To generate 3d random fractal, we first seed a cube defined by its eight vertices and we determine the center. We assign to this point a value $p_{mid}^{(1,8)}$ according to Eq. 5, by using the eight vertices as input. Then, we generate a point at the center of each face, by using the four corners and we determine the value $p_{mid}^{(1,4)}$. Finally, we generate the middle point of each of the 12 cube edges and assign them the value $p_{mid}^{(1,2)}$. The first run of this routine would result in eight subcubes. The second one will result in 64 subcubes. In general, the number of subcubes will be equal to $\left(2^{l}\right)^{d}$, where l is the number of iterations and $d = 3$. A RMDA sponge is shown in Fig. 3.a, with $H = 0.2542$. The values are scaled in such a way that darker areas correspond to higher densities. The average density ρ_m of the sponge has been evaluated by using the initial density $\rho_0 = 917$ kg/m^3 and for several realizations of the fractal. We obtain $\rho_m = 421.99$ kg/m^3 for the snow sponge.



(a) (b)

Fig. 3. (a) Sponge generated by the RMDA with $H = 0.2542$. The values are scaled in such a way that darker areas correspond to higher densities. (b) Log-log plot of σ^2_{DMA} for the 3d sponge generated by 3d RMDA. The data refer to a sponge with $H = 0.2542$ ($D = 2.7458$). The estimated value for full linearity, i.e. the log-log plot of a curve varying as $s^{H_{fit}} = \left[\sqrt{n_1^2+n_2^2+n_3^2}\right]^{2H_{fit}}$, results into $H_{fit} = 0.1854$ (continues line).

To estimate the Hurst exponent of the obtained RMDA sponge, we consider the 3d detrending moving analysis algorithm (DMA) proposed in [11, 12, 13]. The DMA algorithm is based on a generalized high-dimensional variance σ^2_{DMA} around a moving average low-pass filter defined by:

$$\sigma^2_{DMA} = \frac{1}{\mathrm{N}} \sum_{i_1=n_1-m_1}^{N_1-m_1} \sum_{i_2=n_2-m_2}^{N_2-m_2} \cdots \sum_{i_d=n_d-m_d}^{N_d-m_d} \left[f(i_1,i_2,\ldots,i_d) - \bar{f}_{n_1,n_2,\ldots n_d}(i_1,i_2,\ldots i_d) \right]^2, \tag{7}$$

where $f(i_1,i_2,\ldots i_d) = f(\boldsymbol{i})$ is a fractional Brownian field defined over a discrete d-dimensional domain, with sizes N_1, N_2, ..., N_d. It is $i_1 = 1, 2, \ldots, N_1$, $i_2 = 1, 2, \ldots, N_2$ and $i_d = 1, 2, \ldots, N_d$. $\boldsymbol{n} = (n_1, n_2, \ldots, n_d)$ defines the subarrays v_d of the fractal domain with maximum values $n_{1max} = \max\{n_1\}$, $n_{2max} = \max\{n_2\}$ and $n_{dmax} = \max\{n_d\}$; $m_1 = \mathrm{int}(n_1\theta_1)$, $m_2 = \mathrm{int}(n_2\theta_2)$ and $m_d = \mathrm{int}(n_d\theta_d)$ where θ_1, θ_2 and θ_d are parameters ranging from 0 to 1; $\mathrm{N} = (N_1 - n_{1max})(N_2 - n_{2max})\ldots(N_d - n_{dmax})$. The function $\bar{f}_{n_1,n_2,\ldots n_d}(i_1,i_2,\ldots i_d)$ is given by:

$$\bar{f}_{n_1,n_2,\ldots n_d}(i_1,i_2,\ldots i_d) = \frac{1}{n_1 n_2 \ldots n_d} \sum_{k_1=-m_1}^{n_1=1-m_1} \sum_{k_2=-m_2}^{n_2=1-m_2} \cdots \sum_{k_d=-m_d}^{n_d=1-m_d} f(i_1-k_1, i_2-k_2, \ldots i_d-k_d), \tag{8}$$

which is an average of f calculated over the subarrays v_d. By taking $d = 3$, Eq. 7 allows to estimate the Hurst exponent H of the 3d RMDA sponge above described, as seen in Fig. 3.b.

Conclusions

In this paper we have modelled snow granularity by means of different fractal models. Most properties of snow are defined as functions of the specific density ρ_{snow}/ρ_{ice} [1, 2]. The multiscale character of snow, its density and porosity, have been analyzed by the deterministic Menger sponge model. This model allows to define the fractal dimension D as a measure of the space-filling properties of snow, which can be related to the specific density. Moreover, in order to capture the randomness of the local structure of a real snow sample, we have introduced two 3d random fractal models. With these models we were able to investigate how the local structure changes according to a given fractal dimension. Different grain arrangements and cell densities have been explored by varying the fractal dimension.

These models are the first step towards the investigation of the scaling properties of snow in a fully 3d fractal framework and will be relevant to validate experimental results reported in the literature [14, 15].

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