Low-temperature properties of fractional statistics

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Abstract

We consider the low-temperature thermodynamic and magnetic properties of an ideal gas of particles obeying a generic fermion-like fractional statistics. The coefficients of the Sommerfeld expansion are calculated in terms of the central moments of the derivative of the density of entropy with respect to the occupational number.

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1. Introduction

In the recent years, the scientific community has shown a renewed interest towards the fractional statistics. This is mainly related to the application of fractional spin and statistics to the theory of the high T_c superconductivity, of the fractional quantum Hall effect and of the mechanism of the spin and charge carrier separation in quantum antiferromagnets [1–5]. In order to gain a better understanding of the behavior of such systems, great attention has been devoted to the observables strongly dependent on the statistics. Several papers [6–8] dealing with the thermodynamic and magnetic properties have thus appeared besides the wide literature centered around first-principle treatment of quantum mechanics and field theory for system with fractional spin and statistics [9]. In particular, since the deviations from a pure bosonic or fermionic behavior become more evident at low temperatures a considerable effort has been addressed to the investigation of the low-temperature thermodynamic properties of such particles [10–12].

The fractional statistics can be roughly divided in two classes: the boson-like, admitting the possibility that a site can host an arbitrary number of particles, and the fermion-like, for which the number of particles is always subjected to a constraint.

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Let the following discussion be restricted to fermion-like particles. The first paper, where a statistics interpolating between the Fermi and the Bose one appears, is dated 1940 [13]. There, Gentile, defining properly the N-fold occupancy of a site, obtained the para-fermi statistics where the occupation number is given by

$$n = \frac{1}{\exp(\varepsilon) - 1} - \frac{p + 1}{\exp[(p + 1)\varepsilon] - 1} \qquad (1)$$

with $\varepsilon = \beta(\varepsilon - \mu)$. Recently, the parafermion statistics has been adopted to study Fröhlich's one-dimensional superconductors [14]. About a decade later, Green [15], in the framework of the second quantization, analyzed a class of intermediate statistics by generalizing the commutation relations for the creation and annihilation operators. The simplest generalization of the commutation relations is given by

$$a_i a_j^+ - q a_j^+ a_i = \delta_{ij} , \qquad (2)$$

where $-1 \leqslant q \leqslant 1$. The Fermi and the Bose statistics are obtained, respectively, for $q = -1$ and $q = 1$, while other values of q yield the quon statistics [7,16,17]

$$n = \frac{1}{\exp(\varepsilon) - q} , \qquad (3)$$

to which corresponds a density of entropy

$$S(n) = \frac{1}{q}(1 + qn)\log(1 + qn) - n\log n . \qquad (4)$$

In addition to the commutation relation (2), other forms of generalization have been proposed leading to more general quon statistics [7]. A different approach was followed by Haldane [18] . The Haldane fractional statistics was obtained resizing the single-particle Hilbert space available for the nth particle proportionally to the number of added particles. The Haldane generalized exclusion principle can be written as $\Delta d(N) = -g\Delta N$ where $d(N)$ is the dimension of the single-particle Hilbert space when $N - 1$ states are occupied and g is the exclusion statistics parameter, $0 \leqslant g \leqslant 1$ ($g = 1$ for fermions and $g = 0$ for bosons). An explicit realization of the Haldane statistics is the Calogero–Sutherland–Moser system [19,20], for which the paramer g can take any value from 0 to ∞. The density of entropy for the Haldane statistics is given by

$$S(n) = [1 + (1 - g)n]\log\frac{1 + (1 - g)n}{n}$$

$$-(1 - gn)\log\frac{1 - gn}{n} . \qquad (5)$$

The occupation number is implicitly defined through the following relationship [6,21,22]:

$$(1 - gn)^g[1 + (1 - g)n]^{1-g} = ne^\varepsilon . \qquad (6)$$

A great number of papers deal with the thermodynamic properties of systems obeying the Haldane statistics [10–12,23,24]. Many of these properties are common to the whole class of the fermion-like statistics, while others are typical of the particular statistics considered.

The goal of the present paper is to investigate the low-temperature magnetic and thermodynamic properties common to all the fermion-like fractional statistics. We shall focus our attention on the analytical expression of the Sommerfeld series for the chemical potential, the internal energy and the Pauli paramagnetic susceptibility. In particular, since the difference between the statistical and the purely thermodynamic interpretation manifests itself in the existence of fluctuation processes, we shall show the link between the coefficients of the Sommerfeld series and the cumulant expansion. To the best of our knowledge this has not yet been evidenced in literature for fractional statistics.

2. Generalized Sommerfeld expansion

Let us consider a set of N free identical particles with total energy E occupying a group of G states. Let N_i be the number of particles of the ith species, with energy ε_i, occupying G_i states. It results in $\sum_i N_i = N$, $\sum_i G_i = G$, $\sum_i N_i \varepsilon_i = E$. For boson-like particles each of the G_i states can arrange more than one particle, while for fermion-like behavior the ith state contains at most one particle. For particles having an intermediate behavior between bosons and fermions, the mechanism according to which the N_i particles occupy the G_i states is different, resulting in a intermediate statistics.

Let us now consider the system in the thermodynamic limit. The average occupation number for the ith species is given by $n_i = N_i/G_i$. Furthermore, we shall adopt a dispersion relationship $\varepsilon = \varepsilon(k)$ where k is the modulus of the vector wave number. The sum over the index i, reported above, can be transformed into an integration in $d^D k$ where D is the dimension of the space. The density of states $\Omega(\varepsilon)$ is given by the relationship $\Omega(\varepsilon)d\varepsilon = [V/(2\pi)^D]\sigma k^{D-1}dk$, where V is the volume of the system and $\sigma = 2\pi^{D/2}/\Gamma(D/2)$ is a number equal to the area of a sphere with unitary radius in a D-dimensional space. The number N and the energy E of the particles will be given by

$$N = \int_0^\infty \Omega(\varepsilon)n(\varepsilon)\,d\varepsilon\,, \tag{7}$$

$$E = \int_0^\infty \varepsilon\Omega(\varepsilon)n(\varepsilon)\,d\varepsilon\,. \tag{8}$$

Before introducing the Pauli susceptibility χ, a further distinction among the intermediate statistics is required. For physical systems in two spatial dimensions, the wave function for a N-particle system under the exchange of the position of any two particles can be written as $\Psi_{ij} = \exp(2\pi i\alpha)\Psi_{ji}$ where the statistics parameter α can take arbitrary real values, and is equal to 0 or $\frac{1}{2}$, respectively for symmetric wave functions (bosons) and antisymmetric wave functions (fermions) [25]. A fractional electron charge, characterising a quasiparticle which obeys a fractional statistics, has been invoked in several

physical situations such as the fractional Hall effect and the high T_c superconductivity. A different type of intermediate statistics are those describing particles having integer or half-integer spin but satisfying a generalized exclusion principle (parafermion statistics). For both cases, in the framework of the independent particle approximation and neglecting the orbital response, i.e. considering the fractional particle to carry only a spin magnetic moment, the generalized Pauli susceptibility χ defined for particles with two spin degrees of freedom, following Ref. [26], can be obtained as

$$\chi = 2\mu_B^2 \int_0^\infty \frac{d\Omega(\varepsilon)}{d\varepsilon} n(\varepsilon) \, d\varepsilon \,, \tag{9}$$

where μ_B is the Bohr magneton. The average occupation number is obtained by maximizing the entropy S of the system:

$$S = \int_0^\infty \Omega(\varepsilon) \mathcal{S}(n) \, d\varepsilon \,. \tag{10}$$

Keeping fixed the total number N of particles and the total energy E of the system by using the standard Lagrange multiplier method, the following variational problem has to be solved:

$$\frac{\delta}{\delta n} \left(S - \beta E + \beta \mu N \right) = 0 \,, \tag{11}$$

where $\delta/\delta n$ indicates the functional derivatives with respect to the function $n(\varepsilon)$. We obtain

$$\frac{\partial \mathcal{S}(n)}{\partial n} = \varepsilon \,; \tag{12}$$

Eq. (12) defines $n(\varepsilon)$ when $\mathcal{S}(n)$ is known.

Let m be the maximum value of the occupation number $n(\varepsilon)$ with $0 < m < \infty$. This means that the ith state can host a number of particles lower than mG_i. The constraint $0 \leqslant n \leqslant m$ implies a generalization of the Pauli exclusion principle. The occupation probability for the state with energy ε will be $n(\varepsilon)/m$, moreover, the non-occupation probability of the state of energy ε is given by $n_h(\varepsilon)/m$ and $n(\varepsilon) + n_h(\varepsilon) = m$ holds. Furthermore, if a dispersion relationship $\varepsilon(k) = ak^b$ is assumed [10], the density of states can be written as $\Omega(\varepsilon) = \gamma \varepsilon^{d-1}$ where $d = D/b$ and $\gamma = V\sigma/(2\pi)^D ba^d$.

Here we shall consider the low-temperature properties of such a system. At $T = 0$ the system presents a pseudo-fermi surface defined as $n(\varepsilon) = m\theta(\varepsilon_F - \varepsilon)$, and the number of particles is conserved. From Eq. (7) we have $N = \gamma m \int_0^{\varepsilon_F} \varepsilon^{d-1} \, d\varepsilon$. The Fermi energy $\varepsilon_F = \mu_0$ is $\varepsilon_F = ak_F^b$, where $k_F = 2\pi(\rho D/m\sigma)^{1/D}$. Analogously from Eqs. (8) and (9), at $T = 0$, the ground-state energy E_0 and susceptibility χ_0 are given, respectively, by $E_0 = N\varepsilon_F d/(d + 1)$ and $\chi_0 = 2dN\mu_B^2/\varepsilon_F$. Therefore, the $T = 0$ properties depend only on the maximum value m of the occupation number, but are independent of the form of the statistics. In particular, it results that the zero temperature values E_0 and χ_0 for a system of particles obeying intermediate statistics are, respectively, lower and

higher than for fermions. The generalized exclusion principle is indeed more effective than the Pauli exclusion principle to gather the particles at lower energy states. On the contrary, the response to the application of an external magnetic field is higher than that of fermions; the generalized exclusion principle results to be far less effective than the Pauli one in suppressing the tendency of the spin to align with the magnetic field.

The Sommerfeld expansions at $T = 0$ for $\mu(T)$, $E(T)$ and $\chi(T)$ are given by [27]

$$\left(\frac{\mu_0}{\mu}\right)^{d+1} = 1 + d \sum_{j=1}^{\infty} M_j \frac{1}{j} \binom{d-1}{j-1} \left(\frac{kT}{\mu}\right)^j , \tag{13}$$

$$\frac{E}{E_0} = \left(\frac{\mu}{\mu_0}\right)^{d+1} \left[1 + (d+1) \sum_{j=1}^{\infty} M_j \frac{1}{j} \binom{d}{j-1} \left(\frac{kT}{\mu}\right)^j \right] , \tag{14}$$

$$\frac{\chi}{\chi_0} = \left(\frac{\mu}{\mu_0}\right)^{d-1} \left[1 + (d-1) \sum_{j=1}^{\infty} M_j \frac{1}{j} \binom{d-2}{j-1} \left(\frac{kT}{\mu}\right)^j \right] . \tag{15}$$

The quantities M_j are defined by:

$$M_j = -\frac{1}{m} \int_{-\infty}^{+\infty} \varepsilon^j \frac{\partial n(\varepsilon)}{\partial \varepsilon} d\varepsilon , \tag{16}$$

and limiting the integration variable range to positive values, one has

$$M_j = \frac{j}{m} \int_0^{+\infty} \varepsilon^{j-1} \left[n(\varepsilon) + (-1)^j (m - n(-\varepsilon))\right] d\varepsilon . \tag{17}$$

We observe that the quantities M_j are related to the coefficients C_j introduced in Ref. [10] by means of $M_j = j C_{j-1}/m$.

3. Coefficients of the expansion

Let us change the integration variable from the energy to the occupational number $0 \leqslant n \leqslant m$ in Eq. (16). This transformation will yield two results. The first consists in a simplification of the calculation of the quantities M_j, when the statistics is implicitly defined. The Haldane statistics given by Eq. (6) is an example of an implicitly defined statistics. In this case, the calculation of M_j is not easy to be performed by using Eqs. (16) and (17) as observed in Refs. [10,12].

The second is that it allows to understand better the meaning of the quantities M_j. In particular, it will be shown that the coefficients of the Sommerfeld series are related to the fluctuations of the derivative of the density of entropy which corresponds according to Eq. (12) to the energy $\varepsilon(n)$ of a state whose occupation number is n. At first we define the average value $\langle F \rangle$ of a stochastic function $F(n)$ as

$$\langle F \rangle = \frac{1}{m} \int_0^m F(n) dn , \tag{18}$$

where the stochastic variable n is uniformly distributed in the range $0 \leqslant n < \leqslant m$. It is straightforward to observe that the quantity M_j is given by the following relationship:

$$M_j = \left\langle \left(\frac{\partial S}{\partial n}\right)^j \right\rangle \tag{19}$$

that corresponds to the jth moment of the quantity $\partial S(n)/\partial n$. We can also introduce the jth central moment through

$$C_j = \left\langle \left(\frac{\partial S}{\partial n} - \left\langle \frac{\partial S}{\partial n} \right\rangle\right)^j \right\rangle . \tag{20}$$

The central moments C_j are related to the moment M_j by the following relationship:

$$C_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} M_1^{j-i} M_i . \tag{21}$$

From Eq. (13) we can obtain a polynomial relationship approximating $\mu(T)$ when $T \to 0$. By introducing the quantity $\mathcal{T} = kT/\mu_0$ and neglecting the $0(\mathcal{T}^5)$ we have

$$\frac{\mu}{\mu_0} \simeq 1 - M_1\mathcal{T} - (d-1)C_2\frac{\mathcal{T}^2}{2} - (d-1)(d-2)C_3\frac{\mathcal{T}^3}{6}$$

$$-(d-1)(d-3)[(d-2)C_4 - 3(d-1)C_2^2]\frac{\mathcal{T}^4}{24} . \tag{22}$$

For what concerns the quantities E and χ at $T \to 0$, taking into account Eqs. (14), (15) and Eq. (22), one has

$$\frac{E}{E_0} \simeq 1 + (d+1)C_2\frac{\mathcal{T}^2}{2} + (d+1)(d-1)C_3\frac{\mathcal{T}^3}{3}$$

$$+(d^2 - 1)[(d-2)C_4 - 3(d-1)C_2^2]\frac{\mathcal{T}^4}{8} , \tag{23}$$

$$\frac{\chi}{\chi_0} \simeq 1 - (d-1)C_2\frac{\mathcal{T}^2}{2} - (d-1)(d-2)C_3\frac{\mathcal{T}^3}{3}$$

$$-[(d-1)(d-2)(d-3)C_4 - (d-1)^2(3d-7)C_2^2]\frac{\mathcal{T}^4}{8} . \tag{24}$$

The specific heat $C_V = \partial E/\partial T$ becomes

$$\frac{C_V}{Nd} \simeq C_2\mathcal{T} + (d-1)C_3\mathcal{T}^2$$

$$+(d-1)[(d-2)C_4 - 3(d-1)C_2^2]\frac{\mathcal{T}^3}{2} . \tag{25}$$

It can be observed that the quantities $\kappa_1 = M_1$, $\kappa_2 = C_2$, $\kappa_3 = C_3$, $\kappa_4 = C_4 - 3C_2^2$ are the first four coefficients of the cumulant expansion [28,29].

When $d = 1$, i.e., for example, if the system is planar and the dispersion relation is quadratic, i.e., $\varepsilon \propto k^2$, then $\mu(T) = \mu_0$ since the dominant asymptotic expansion vanishes. Obviously, this does not mean that $\mu(T)$ is independent of T, but simply that in the Sommerfeld series, the subdominant terms z^{-l} do not appear (where $z = \exp(\beta\mu)$ is the fugacity and $l > 0$) [12]. This occurs also for χ, while for E we observe that it assumes a quadratic behavior and thus C_V becomes linear.

In the following we shall analyze separately some of the coefficients of the Sommerfeld expansion given by Eqs. (19) and (20), which depend on the statistics and in general on m. The first coefficient is given by $M_1 = [S(m) - S(0)]/m$. If we take into account the definition of the statistical weight $\mathcal{W}$ given by $S(n) = \ln \mathcal{W}(n)$, the previous relation indicates that $\mathcal{W}(m)$ is related to $\mathcal{W}(0)$ by $\mathcal{W}(m) = \mathcal{W}(0)\exp(mM_1)$. On this account, if $M_1 = 0$ as it occurs for the Fermi, Haldane and para-fermi distributions, the maximum and the zero occupation states are equiprobable. Thus, M_1 is strongly affected by the symmetry of the distribution $n(\varepsilon)$. Similar considerations were obtained in Ref. [30] using a path-integral realization for systems with exclusion statistics. Moreover, if $M_1 = 0$ the dependence of μ on T is quadratic when $T \to 0$. Conversely, when $M_1 \neq 0$, $\mu(T)$ is linear for $T \to 0$. This occurs for the quon statistics where $m = -1/q$, $M_1 = -\ln m$ and then $\mathcal{W}(m) = m^{-m}$.

For what concerns the second central moment C_2, it is the variance of $\partial S(n)/\partial n$. It is positive and distinct from zero and determines the behavior at low temperature of all the properties of the particle gas. For the Haldane statistics having $m = 1/g$, $C_2 = \pi^2/3m$, and for the parafermion particles where $m = p$, $C_2 = 2\pi^2/3(m + 1)$. By the dependence of the coefficients C_2 on m we can deduce that the modulus of the relative variation of both the energy and the Pauli susceptibility, when the temperature varies around $T = 0$, decreases as m increases. On the contrary, for the quons where $m = -1/q$ we have that C_2 is independent of m and is equal to the value $\pi^2/3$ as in the case of fermions.

The central moments C_j of higher order are less important in determining the behavior with T for the above-described quantities. For the Haldane statistics in Ref. [12] it is shown that for $j \geqslant 0$, $C_j = mC_{j+1}/(j + 1)$ are polynomials in g of order $j - 1$ and are calculated for $j = 1, 2, 3$. For the para-fermi statistics it is possible to calculate all the central moments C_j obtaining the following results:

$$C_{2j+1} = 0, \quad j \geqslant 0$$

$$C_{2j} = (2\pi)^{2j}|B_{2j}|\frac{1}{p}\left[1 - \frac{1}{(p+1)^{2j-1}}\right], \quad j > 0, \tag{26}$$

where B_l are the Bernoulli numbers.

4. Conclusions

As a conclusion we recall the results obtained here. We have derived the Sommerfeld polynomial expansion up to the T^4 order for an ideal gas of particles obeying to

a generic fractional fermion-like statistics. The coefficients of the polynomials obtained depend on the particular statistics and can be calculated starting from the central moments of the first derivative of the density of entropy $\partial S(n)/\partial n$. Finally, in principle, we can devise a realistic experiment that, starting from the measurement at $T = 0$ of the susceptibility χ_0, can yield the value of m . By using this value of m, the statistics can be selected by analyzing the behavior of C_V when $T \to 0$.

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