Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Physica A 340 (2004) 544–551
PHYSICA A
www.elsevier.com/locate/physa

Directed self-organized critical patterns emerging from fractional Brownian paths

Anna Carbone[a,*], H. Eugene Stanley[b]

[a] *Dipartimento di Fisica, INFM—Inst. Nazionale di Fisica della Mat., Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy*
[b] *Department of Physics, Center for Polymer Studies, Boston University, Boston, MA 02215, USA*

Received 8 March 2004

Abstract

We discuss a family of clusters $\mathscr{C}$ corresponding to the region whose boundary is formed by a fractional Brownian path $y(i)$ and by the moving average function $\widetilde{y}_n(i) \equiv \frac{1}{n}\sum_{k=0}^{n-1} y(i-k)$. Our model generates *fractal directed patterns* showing spatio-temporal complexity, and we demonstrate that the cluster area, length and duration exhibit the characteristic scaling behavior of SOC clusters. The function $C_n(i)$ acts as a *magnifying lens*, zooming in (or out) the 'avalanches' formed by the cluster construction rule, where the *magnifying power* of the zoom is set by the value of the amplitude window n. On the basis of the construction rule of the clusters $C_n(i) \equiv y(i) - \widetilde{y}_n(i)$ and of the relationship among the exponents, we hypothesize that our model might be considered to be a generalized stochastic directed model, including the Dhar–Ramaswamy (DR) model and the stochastic models as particular cases. As in the DR model, the growth and annihilation of our clusters are obtained from the set of intersections of two random walk paths, and we argue that our model is a variant of the directed self-organized criticality scheme of the DR model.

PACS: 05.40. − a; 05.65.+b; 05.50.+q

Keywords: Self-organized criticality; Fractional Brownian paths; Critical scaling exponents

Random processes are easily described for systems where the number of basic random components becomes large, i.e., their size is negligible and thus interaction among

* Corresponding author. Tel.: +39-011-564-7378; fax: +39-011-564-7399.
E-mail address: acarbone@polito.it (A. Carbone).

0378-4371/$ - see front matter © 2004 Published by Elsevier B.V.
doi:10.1016/j.physa.2004.05.004

them can be disregarded. In real extended dynamical systems, decorrelation does not hold and the evolution occurs through a critical behavior where interaction is strong and statistical scaling properties are highly nontrivial. A breakthrough in the understanding and description of long-range space-time correlation is represented by the self-organized criticality (SOC) model proposed in Refs. [1,2] and in the literature that consequently flourished [3–20].

The SOC model has demonstrated the ability to describe time–space correlated evolution of several critical phenomena as interface depinning [8], the Bak–Sneppen evolution model [9], the earthquake model [10] and the forest-fire model [11]. The emergence of spatio-temporal complex patterns, initiating at extremal sites rather than at randomly chosen sites, has been furthermore established as a characteristic signature of self-organized critical evolution [12,13,15].

Sandpile models are cellular automata (CA) with an integer or continuous variable z_i defined on a d-dimensional lattice of size L . At each time step a particle (or energy) is added to a randomly chosen site, until the variable z_i, which denotes the number of grains (or the energy) at site i reaches the threshold value z_c. When this occurs the site "relaxes", i.e.,

$$z_i \to z_i - z_c \tag{1}$$

and particles are isotropically transferred to the nearest neighbors

$$z_{i'} \to z_{i'} + y_{i'} \ . \tag{2}$$

The relaxation of a site can induce a number of other sites to relax in turn if, because of the particles received, they exceed the threshold. From the moment a site topples, the addition of particles stops until all sites have relaxed ($z_i < z_c$ for all i). This condition assures that the driving force is 'slow' being the driving time exceedingly longer than the characteristic time of toppling instances. The sequence of toppling events during this interval constitutes an avalanche. For conservative models, the number of transferred particles equals the number of particles lost by the relaxing site ($\sum y_j = z_c$) and dissipation occurs only at boundary, from which particles can escape the system. Under these conditions the system reaches a stationary state characterized by a sequence of avalanches. Since the SOC algorithm is implemented basically as a cellular automaton, the cluster growth is intrinsically of diffusive nature [21–23].

The total number of toppling sites s, the avalanche diameter l, and the avalanche lifetime τ are usually used to study the dynamics underlying the avalanche. The quantities s, l and τ are related by power law: $s \sim \tau^{D_s/z}$ and $l \sim \tau^z$, D_s and z are, respectively, the avalanche dimension exponent and the dynamical exponent.

The pdf for s, l and τ scale, respectively, as

$$P(s) \sim s^{-\tau_s} f(s/s_c) \ , \tag{3}$$

$$P(l) \sim l^{-\tau_l} f(l/l_c) \ , \tag{4}$$

$$P(\tau) \sim \tau^{-\tau_t} f(\tau/\tau_c) \ . \tag{5}$$

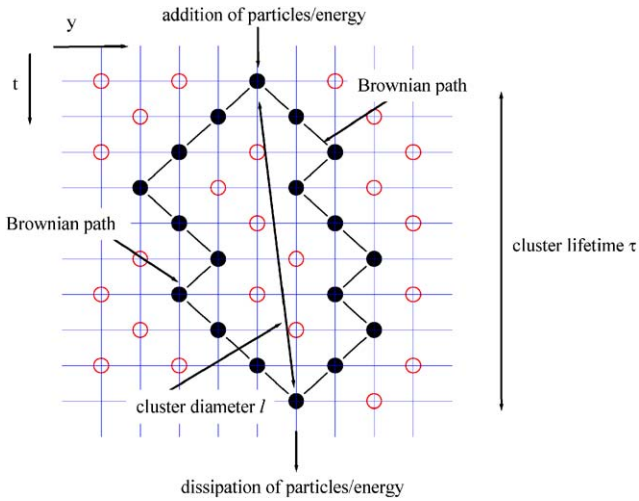

Fig. 1. Directed sandpile models on a lattice with finite L. Grains from the active sites in the row t topple onto sites in row $t+1$.

Using the above set of equation, one obtains the following relationships among the SOC exponents

$$\tau_t = z\tau_l \ , \tag{6}$$

$$(\tau_t - 1)z = (\tau_s - 1)D_s \ . \tag{7}$$

In its simplest version, the Dhar–Ramaswamy (DR) directed sandpile model [6,7] can be described assuming that the system is driven by particles added at the top layer $i=0$ and removed from the bottom layer $i=L$ (see Fig. 1). The preferred direction is achieved using still Eq. (1), but adopting instead of Eq. (2), the following:

$$z_{i'} \to z_{i'} + y_{i'}(i' > i) \ . \tag{8}$$

The DR cluster is schematically represented in Fig. 1. As has been observed in Ref. [6], the cluster corresponds to the region formed by the path of two annihilating random walkers starting at the site where the cluster initiates. In this sense the DR cluster are analogous to the directed percolation clusters of Domany and Kinzel [24–26]. The DR model admits analytical solution in every dimension. For the two-dimensional case, the cluster exponents are $z=1$, $D_s = \frac{3}{2}$ $\tau_l = \tau_t = \frac{3}{2}$, $\tau_s = \frac{4}{3}$.

Other interesting variants of the original model of Bak et al. are the stochastic models [14–17]. The essential feature of the Manna model is that the toppling is replaced by a 'reaction' at a given site i followed by a random reallocation of scattered particles among the neighbors. Recently, Dhar has demonstrated that the Manna sandpile model satisfies the Abelian property [18]. Two variants, known as stochastic directed models (SDM) of the Dhar–Ramaswamy model have been proposed in Ref. [17]. Where two different stochastic toppling rules, the exclusive and the non-exclusive, are introduced in the DR model. The values of the cluster exponents are $\tau_s = 1.42$, $D_s = 1.72$, $\tau_t = 1.70$

for the exclusive SDM and $\tau_s = 1.43$, $D_s = 1.75$, $\tau_t = 1.70$ for the non-exclusive SDM. The authors conclude that the SDM and the DR model belong to different universality classes. In general, for stochastic models, each site is characterized by an infinite stack of number that according to some simple stochastic rules address the grains towards a particular set of sites.

In the present work, we propose a family of stochastic directed clusters generated by fractional Brownian paths with different correlation properties. We show that the cluster area, length and duration exhibit the characteristic scaling behavior of SOC clusters. We calculate 'exactly' the characteristic exponents z, D_s τ_l, τ_t, τ_s in the two-dimensional case for any value of the correlation exponent of the generalized fractional Brownian motion: $z = 1$, $D_s = 1 + H$, $\tau_l = \tau_t = 2 - H$; $\tau_s = 2/(1 + H)$. The previous values, for the case of uncorrelated Brownian motion (the simple random walk $H = 0.5$) coincide with the exponents of the DR model. According to our model, with the toppling rules of Ref. [17], a random walk with correlation exponent $H < 0.5$, i.e., with a path variance varying with time less than in a fully uncorrelated Brownian motion, might account for the values of the exponents.

To achieve our goal, we use a generalized Brownian walk $y(i)$ defined by $y(i) \equiv \sum_{k=0}^{i-1} \xi_k$, where the steps ξ_k are taken from a discrete Gaussian process with $\langle \xi_k \rangle = 0$ and $\langle \xi_k^2 \rangle = \sigma$ and the brackets $\langle \cdots \rangle$ denote the ensemble average. The mean square displacement of $y(i)$ scales with Δi as $\langle [y(i)]^2 \rangle \sim (\Delta i)^{2H}$, where H is the Hurst exponent ($0 < H < 1$). The moving average function $\widetilde{y}_n(i)$ is

$$\widetilde{y}_n(i) \equiv \frac{1}{n} \sum_{k=0}^{n-1} y(i-k) \tag{9}$$

which is a linear operator whose output is still a generalized Brownian motion, but now with the high-frequency components of the signal averaged out according to the window amplitude n [28].

In order to characterize the *clusters* $\mathscr{C}$ corresponding to the regions bounded by $y(i)$ and $\widetilde{y}_n(i)$ in terms of the characteristic exponents of SOC systems, we define for each cluster the following quantities (Fig. 2 and Fig. 3):

(i) *Cluster length* ℓ_j:

$$\ell_j \equiv \sum_{i=i_c(j)}^{i_c(j+1)} \delta\ell(i)\,, \tag{10}$$

(ii) *Cluster lifetime* τ_j:

$$\tau_j \equiv i_c(j+1) - i_c(j)\,, \tag{11}$$

(iii) *Cluster area* s_j:

$$s_j \equiv \sum_{i=i_c(j)}^{i_c(j+1)} |y(i) - \widetilde{y}_n(i)| \Delta i\,, \tag{12}$$

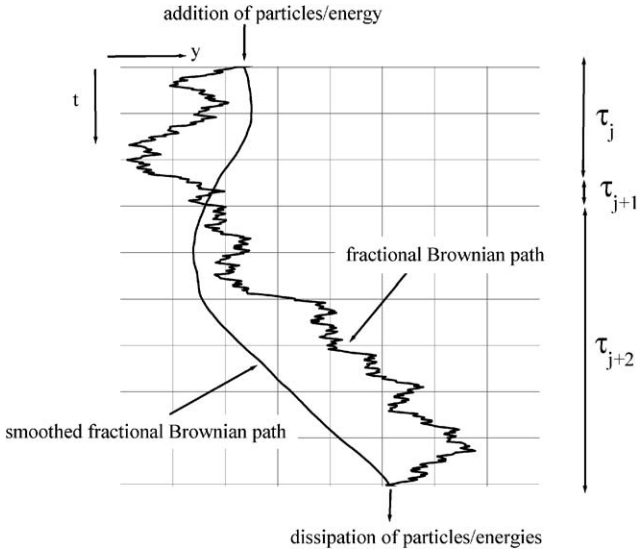

Fig. 2. Directed sandpile models on a infinite lattice. The functions $y_n(i)$ and $\widetilde{y}_n(i)$ are shown. In this figure the grid does not coincide with the lattice where the instances of toppling take place.

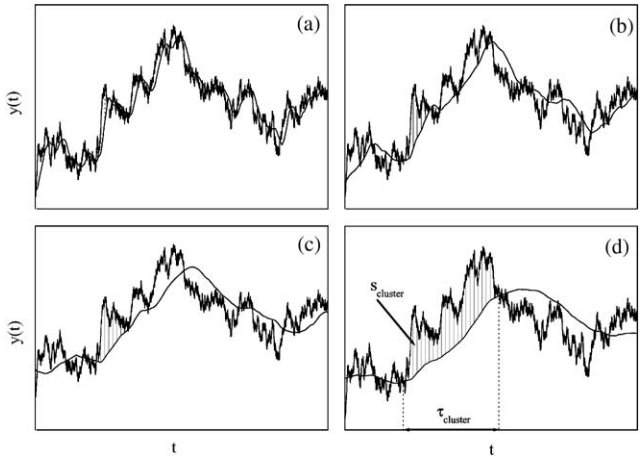

Fig. 3. The signal $y(t_i)$ for the case $H = 0.5$ and the moving average function $\widetilde{y}_n(t_i)$ shown with four different averaging box sizes, (a) $n = 200$, (b) 600, (c) 1000, (d) 2000. The shaded area in (d) represents a typical cluster.

where the index j refers to each cluster, $i_c(j)$ and $i_c(j+1)$ are the values of the index i corresponding to two subsequent intersections between $\widetilde{y}_n(i)$ and $y(i)$ and Δi is the elementary time interval corresponding to each step of the random walker.

Let $\ell \equiv \langle \ell_j \rangle_\tau$ and $s \equiv \langle s_j \rangle_\tau$ indicate the average value respectively of the cluster length and area over the ensemble $\mathscr{C}_\tau$ of the clusters having duration τ. Log–log plots of the cluster length ℓ and the cluster area s plotted against the cluster lifetime τ for long-range correlated time series are consistent with the following power-law relationships:

$$\ell \sim \tau^{\psi_\ell} \quad [\psi_\ell = 1] \tag{13}$$

and

$$s \sim \tau^{\psi_s} \quad [\psi_s = 1 + H] \,. \tag{14}$$

Eq. (13) follows if the relationships $\Delta y(i) \sim (\Delta i)^H$ and $\Delta l(i) = \sqrt{\Delta y(i)^2 + \Delta i^2}$ are taken into account to calculate the length $\Delta l(i)$ [Eq. (10)]. Eq. (14) follows if the relationship:

$$\widetilde{y}_n(i) - y(i) \approx D_H \frac{\Delta i}{(\Delta n)^2} \nabla^2 y(i) \tag{15}$$

is taken into account [23], where D_H is the generalized diffusion coefficient for fractional Brownian motion, and the other quantities have the usual meaning. The term on the right side of Eq. (15) is proportional to the average displacement of the random walker Δi and thus varies as $(\Delta i)^H$. Using Eq. (15) to calculate the sum of terms $\widetilde{y}(i) - y_n(i)$ over the time interval $\tau_j \equiv i_c(j+1) - i_c(j)$ [Eq. (12)], the relation $\psi_s = 1 + H$ follows.

Next we calculate the pdf $P(\tau)$ of the cluster lifetime τ. Our numerical results are consistent with power-law behavior:

$$P(\tau) \sim \tau^{-\beta}, \quad [\beta = 2 - H] \,, \tag{16}$$

where $P(\tau)$ is the first return pdf [27,12] of the crossing points between the signal $y(i)$ and the moving average function $\widetilde{y}_n(i)$. Therefore Eq. (16) can be derived from the relation $\langle \tau \rangle \equiv N_{\max}/N_\times$ between the mean time interval $\langle \tau \rangle$ and the total number of crossing points $N_\times$. To express $\langle \tau \rangle$ in terms of $N_{\max}$, we write $\langle \tau \rangle \equiv \int_1^{N_{\max}} \tau P(\tau) d\tau \sim N_{\max}^{2-\beta}$. Similarly, we can express $N_\times$ in terms of $N_{\max}$ as $N_\times \sim N_{\max}^{1-H}$, which follows from the fact that the fractal dimension of the set of crossing points is $1 - H$ (the co-dimension of the set of crossing points $d - d_\times$ is equal to the sum of the co-dimension $2 - H$ of the signal and the co-dimension $2 - 1$ of the moving average). Thus $2 - \beta = 1 - (1 - H)$.

The pdf $P(\tau)$ with Eqs. (13) and (14), allows us to calculate the pdfs $P(\ell)$ and $P(s)$:

$$P(\ell) = P(\tau(\ell)) \frac{d\tau}{d\ell} \sim \ell^{-\alpha}, \quad [\alpha = 2 - H] \,, \tag{17}$$

$$P(s) = P(\tau(s)) \frac{d\tau}{ds} \sim s^{-\gamma}, \quad [\gamma = 2/(1 + H)] \,. \tag{18}$$

Eq. (17) follows from the reasoning used above, while Eq. (18) follows on using Eq. (14), allowing γ to be expressed in terms of β and ψ_s:

$$\gamma = \frac{\beta + 1 - \psi_s}{\psi_s} = \frac{2}{1 + H} \,. \tag{19}$$

All five new exponents ψ_ℓ, ψ_s, α, β, γ satisfy relations (6) and (7). In particular, we have for the dynamic exponent

$$z = \psi_\ell = 1 \tag{20}$$

and for the avalanche exponent size

$$D_s/z = \psi_s = 1 + H \ . \tag{21}$$

For the pdf exponents we find the analogs of Eqs. (3)–(5) to be

$$\tau_l = \alpha = 2 - H \ , \tag{22}$$

$$\tau_t = \beta = 2 - H \ , \tag{23}$$

$$\tau_s = \gamma = \frac{2}{(1+H)} \ . \tag{24}$$

In summary, we find 'exact' expressions for the exponents z, D_s τ_l, τ_t, τ_s in the two-dimensional case for any value of the correlation exponent H of the generalized fractional Brownian motion: $z = 1$, $D_s = 1 + H$, $\tau_l = \tau_t = 2 - H$; $\tau_s = 2/(1 + H)$. The previous exponents, for the case of uncorrelated Brownian motion (the simple random walk $H = 0.5$) coincide with the exponents of the DR model. With the toppling rules of paper [17], a random walk is generated with correlation exponent $H < 0.5$—i.e., with a path variance changing with time less than the fully uncorrelated Brownian motion. This is consistent, according to our picture, with different values of the exponents of the 'stochastic' SDM model compared to the 'deterministic' DR model. Other features of the $\mathscr{C}$ clusters will be treated in a future work.

We thank Bosilika Tadic for discussion. A.C. acknowledges support by the MIUR under the program PRIN2003 *Excess noise in nanoscaled optoelectronic devices*. H.E.S thanks the NIH (Grant 1-R01-HL71972-01) and the NIH/National Center for Research Resources (P41RR13622) for support.

References

[1] P. Bak, C. Tang, K. Wiesenfeld, Phys. Rev. Lett. 59 (1987) 381.
[2] P. Bak, C. Tang, K. Wiesenfeld, Phys. Rev. A 38 (1988) 364.
[3] H.J. Jensen, Self-Organized Criticality: Emergent Complex Behavior in Physical and Biological Systems, Cambridge University Press, Cambridge, 2000.
[4] L.P. Kadanoff, S.R. Nagel, L. Wu, S. Zhou, Phys. Rev. A 39 (1989) 6524.
[5] Y.C. Zhang, Phys. Rev. Lett. 63 (1989) 470.
[6] D. Dhar, R. Ramaswamy, Phys. Rev. Lett. 63 (1989) 1659.
[7] D. Dhar, Phys. Rev. Lett. 64 (1990) 1613.
[8] K. Sneppen, Phys. Rev. Lett. 69 (1992) 3539.
[9] P. Bak, K. Sneppen, Phys. Rev. Lett. 71 (1993) 4083.
[10] K. Christensen, Z. Olami, P. Bak, Phys. Rev. Lett. 68 (1992) 2417.
[11] D.L. Turcotte, Rep. Prog. Phys. 62 (1999) 1377.
[12] S. Maslov, M. Paczuski, P. Bak, Phys. Rev. Lett. 73 (1994) 2162.
[13] M. Paczuski, S. Maslov, P. Bak, Phys. Rev. E 53 (1996) 414.

[14] S.S. Manna, J. Phys. A: Math. Gen. 24 (1991) L363.
[15] B. Tadic, D. Dhar, Phys. Rev. Lett. 79 (1997) 1519.
[16] M. Paczuski, K.E. Bassler, Phys. Rev. E 62 (2000) 5347.
[17] R. Pastor-Satorras, A. Vespignani, Phys. Rev. E 62 (2000) 6195.
[18] D. Dhar, Physica A 270 (1999) 69.
[19] A. Chessa, H.E. Stanley, A. Vespignani, S. Zapperi, Phys. Rev. E 59 (1999) R12.
[20] A. Vespignani, R. Dickman, M.A. Munoz, S. Zapperi, Phys. Rev. E 62 (2000) 4564.
[21] R. Dogaru, Universality and Emergent Computation in Cellular Neural Networks, World Scientific Series in Nonlinear science, World Scientific, Singapore, 2003.
[22] J.P. Boon (Ed.), Lattice Gas Automata: Theory, Simulation, Implementation, J. Stat. Phys. 68 (1992) 87.
[23] J.R. Weimar, J.-P. Boon, Phys. Rev. E 49 (1994) 1749.
[24] E. Domany, W. Kinzel, Phys. Rev. Lett. 47 (1981) 5.
[25] E. Domany, W. Kinzel, Phys. Rev. Lett. 53 (1984) 311.
[26] F.Y. Wu, H.E. Stanley, Phys. Rev. Lett. 48 (1982) 775.
[27] M. Ding, W. Yang, Phys. Rev. E 52 (1995) 207.
[28] A. Carbone, G. Castelli, H.E. Stanley, Phys. Rev. E 69 (2004) 026105.