Space geometry in rotating reference frames: A historical appraisal

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Abstract

The problem of giving a relativistic description of the geometry of a rotating disk has a history nearly as old as that of the theory of relativity itself. Already in 1909 Ehrenfest formulated his famous paradox in the context of the special theory of relativity. A few years later Einstein made heuristic use of this problem in order to motivate the introduction of non-Euclidean geometry in a relativistic theory of gravity. We shall here follow the conceptual evolution of this topic from Ehrenfest and Einstein to the present time. In particular we emphasise the importance of taking the relativity of simultaneity properly into account in order to obtain a full understanding of the issues connected with Ehrenfest's paradox.

Introduction

The relativistic description of the geometry of rotating bodies is more than 90 years long. It started with a short note by Paul. Ehrenfest [1] who pointed out the contradictory conditions that the radius of a relativistically rotating cylinder has to fulfil: On the one hand the periphery is Lorentz contracted, and on the other hand a radial line on the cylinder is not. This problem was soon taken up by among others Max Planck [2] and Albert Einstein [3], and it has been discussed right up to the present day [4].

We shall here follow this discussion and try to see what conceptual difficulties made this topic so long lived.

The discussion of 1910 and 1911 in Physikalische Zeitschrift

In 1909 Ehrenfest [1] wrote that he was trying to understand Max Born's [5] notion of relativistic rigidity. He first pointed out that the notion of rigid motion of a body can be formulated either with reference to an inertial frame in which the body moves, or with reference to the local rest frame of an element of the body. In the first case he writes (translated to English):

> To say that a body remains relativistically rigid means: It deforms continuously by arbitrary motion so that each of its infinitesimal elements Lorentz contracts (relative to its rest length) all the time in accordance with the instantaneous velocity of each of its elements, as observed by an observer at rest.

In the second case:

> Relativistic rigidity means: As measured by a continuum of observers co-moving with each point of an arbitrarily moving body each element of the body remain undeformed.

He then writes:

> Consider a relativistically rigid cylinder with radius R and height H. It is given a rotating motion about its axis, which finally becomes constant. As measured by an observer at rest the radius of the rotating cylinder is R'. Then R' has to fullfill the following two contradictory requirements:
>
> a) The circumference of the cylinder must obtain a contraction $2\pi R' < 2\pi R$ relative to its rest length, since each of its elements move with an instantaneous velocity $R'\omega$.
>
> b) If one considers each element along a radius, then the instantaneous velocity of each element is directed perpendicular to the radius. Hence, the elements of a radius cannot show any contraction relative to their rest length. This means that: $R' = R$.

This contradiction was termed "Ehrenfest's paradox" ("Ehrenfestschen Paradoxon") by V. Varićak [6]. As pointed out by M. J. Klein [7] Ehrenfest's article was the first in a series of critical analyses which demonstrated that the concept of Born rigid motion cannot be applied to rotating motion in general.

Ehrenfest's paradox was discussed by Max Planck in 1910 [2]. He pointed out:

> The statement that the volume of a body with velocity v measured by an observer at rest is less by the ratio $\sqrt{c^2 - v^2} : c$ than its volume measured by a co-moving observer with velocity v, must not be mixed up with another statement, that the volume of a body which is brought from a state of rest to a velocity v is decreased by a factor $\sqrt{c^2 - v^2} : c$. The first statement is one of the fundamental requirements of the theory of relativity, while the last statement is not generally correct.

Planck then argued that the task of specifying the final state of a body set into rotation is a dynamical problem involving the theory of elastic media. It was, however, immediately clear that it is impossible to set a body into rotation while maintaining Born rigidity (see further references on this point in [8].

Then Theodor Kaluza [9] pointed out the necessity of considering non-Euclidean geometry in order to give a relativistic description of the geometry of rotating bodies. He wrote:

> According to the theory of relativity the "proper geometry" (corresponding to a fixed proper time) is in general the geometry of a surface orthogonal to the bundle of world lines of the body. Two events are "simultaneous" if they belong to this surface.

He further said:

> A closer investigation shows that the proper geometry of a rotating disc is a non-Euclidean special Lobachevskian geometry.

However, the details of the investigation were not included in the article.

In response to an article by W. von Ignatowski [10] on relativistic kinematics Ehrenfest [11] offered a gedanken experiment: Consider a disk at rest with equally spaced circles about the origin of the rotational axis engraved on its surface. Let these circles be recorded on a piece of tracing paper by a stationary observer. Assume now that the disk could be put into rotation while remaining Born rigid and then rotate at constant angular velocity about an axis through its centre, while the observer remains at rest. If the observer *instantaneously* registered the rotating disk's markings on another piece of tracing paper, he would find upon comparison with the other piece of paper that the radial lengths are the same, but the circumference measured during rotation is less than before. This contradiction shows that the assumption of Born rigidity is not compatible with putting the disk into rotation.

In an article on relativistic theory of elasticity Ignatowski [12] calculates the change of the radius and the periphery of a disk with given elastic properties that is put in rotational motion. In a critical comment to this work Ehrenfest [13] interprets Ignatowski to mean that the motion is Born rigid. Hence he indicates that the calculation, or at least its physical interpretation, contains an inconsistency.

In February 1911 V.Varićak [6] claimed that according to Einstein's theory the Lorentz contraction is a sort of observational illusion, and that in reality bodies are not contracted when moving. He thus concluded that there is no paradox. Einstein considered this misinterpretation of the theory of relativity to be rather serious and therefore gave an answer [3] where he explained the relativistic meaning of the Lorentz contraction.

Einstein's realisation that the geometry on the rotating disk is non Euclidean

Although Einstein did not participate in the discussion of Ehrenfest's paradox, he was well aware of the problem with Born rigidity as applied to rotating motion, but he was more concerned with the purely geometrical aspect. Working on a relativistic theory of gravitation his discovery of the equivalence of being in a field of gravity and being in a non-inertial reference frame motivated him to search for a geometrical theory of gravity. Thinking about spatial measurements on a rotating disk he arrived at the conclusion that one needed to free one self from the restrictions of the Euclidean geometry.

He made several notes about spatial geometry in a rotating reference frame both in letters and in publications (see J. Stachel [14, 15].) In the great article in 1916 where he presented the general theory of relativity Einstein considered the disk as a rotating reference frame K and imagined this frame filled by radial and tangential standard rods. The definition of a standard rod is that it is Born rigid, so that it gets a Lorentz contraction when moving. Denoting the inertial rest frame of the axis by K he then wrote [16]:

> We suppose that the circumference and diameter of a circle have been measured with a standard measuring rod infinitely small compared with the radius, and that we have the quotient of the two results. If this experiment were performed with measuring rods at rest relatively to the Galilean system K', the quotient would be π. With measuring rods at rest relatively to K, the quotient would be greater than π. This is readily understood if we envisage the whole process of measuring from the "stationary" system K', and take into consideration that the measuring rods applied to the periphery undergoes a Lorentz contraction, while the ones applied along the radius do not. Hence Euclidean geometry does not apply to K.

Einstein has here for the first time made it clear that the length of the periphery of a rotating disk in *longer* than $2\pi r$ not shorter as stated in Ehrenfest's paradox. The reason for this difference is that Ehrenfest considered a hypothetical, but impossible situation where a disc had been put into rotational motion in a Born rigid way, while Einstein considered a situation in which the disk had been put into rotation in an arbitrary way, but the measuring rods were required to be Born rigid.

Einstein gave similar discussions of the geometry of the rotating disc in his semi popular introduction to the general theory of relativity [17] and in *The Meaning of Relativity* [18] based upon his Princeton lectures in 1921. The most complete treatment of this topic by Einstein is in fact found in a letter to Joseph Petzold dated August 19, 1919. This letter was translated by J. Stachel and published in 1989 [14]. In the part of this letter which concerns the geometry of the rotating disk, Einstein writes:

> A rigid circular disk must break up if it is set into rotation, on account of the Lorentz contraction of the tangential fibres and the non-contraction of the radial ones. Similarly, a rigid disk in rotation must explode as a consequence of the inverse changes in length, if one attempts to bring it to the rest state.
>
> Now you believe that a rigidly rotating circular line must have a circumference that is less than $2\pi r$ because of the Lorentz contraction. The basic error here is that you instinctively set the radius r of the rotating circular line equal to the radius r_0 that the circular line has in the case when it is at rest. This however, is not correct; because of the Lorentz contraction rather $2\pi r = 2\pi r_0 \sqrt{1 - v^2 / c^2}$.
>
> The treatment of the metric of the circular disk runs as follows in detail. Let U_0 be the circumference, r_0 the radius of the rotating disk, considered from the standpoint of K' [that is, the rest frame]; then, on account of ordinary Euclidean geometry,
>
> $$U_0 = 2\pi r_0 \tag{1}$$
>
> U_0 and r_0 naturally are to be thought of as measured with non rotating measuring rods, that is, at rest relative to K'.
>
> Now let me imagine co-rotating measuring rods of rest length 1 laid out on the rotating disk, both along a radius as well as the circumference. How long are these considered from K'? Let us imagine, in order to make this clearer to ourselves, a “snapshot” taken from K' (definite time t_0). On this snapshot the radial measuring rods have length 1, the tangential ones, however, the length $\sqrt{1 - v^2 / c^2}$. The “circumference” of the circular disk (considered from the rest frame of the disk, K) is nothing but the number of tangential measuring rods that are present in the snapshot along the circumference, whose length considered from K' is U_0 . Therefore
>
> $$U = U_0 / \sqrt{1 - v^2 / c^2} \tag{2}$$
>
> On the other hand, obviously

$$r = r_0 \tag{3}$$

(since the snapshot of the radial unit measuring rod is just as long as that of a measuring rod at rest relative to K').

Therefore, from (1)-(3),

$$\frac{U}{r} = \frac{U_0}{r_0\sqrt{1 - v^2 / c^2}} = \frac{2\pi}{\sqrt{1 - v^2 / c^2}} \tag{4}$$

Since $v = r\omega$ equation (4) implies that the ratio between the circumference and the diameter gets larger with increasing radius. The position dependence of this ratio can be measured by observers on the disk. Hence, they would conclude that the geometry on the disk is non-Euclidean.

These considerations by Einstein were soon made well known in the book by Born [19] and later by Einstein and Infield [20]. The first edition of Max Born's popular book [19] on the theory of relativity came in 1920. Here he noted an interesting consequence of the non-Euclidean geometry on a rotating disk combined with the principle of equivalence: In a gravitational field a standard measuring rod is longer or shorter according to the position at which it is situated. A. Metz [21] later gave an illustration where he compared the measuring rods on a rotating disk with the wagons of a model train travelling around a circular path, and K. Kraus [22] considered the intervals between measuring rods attached to the spokes of a rotating wheel. However, he did not take the relativity of simultaneity properly into account.

Not everyone agreed with Einstein. J. Becquerel [23] argued in 1922 that the quotient between the periphery and diameter of a rotating disk is *less* than π. He arrived at this by saying that the measurements of the periphery in K is obtained by means of measuring rods at rest in the inertial frame K' as observed from K. The number of measuring rods around the periphery is invariant. If there are *n* measuring rods along the periphery in K', where the quotient between the circumference and the diameter is equal to π, then as observed in K the number of measuring rods is the same, but each rod is Lorentz contracted. Hence an observer in K would say that this measuring procedure leads to a quotient between the periphery and the diameter less than π. This argument is fallacious, however, because a measurement of the length of the circumference of the rotating disk must be performed with standard measuring rods at rest on the disk, not at rest in the inertial rest frame of the axis.

Shu [24] later came to the conclusion that the space geometry on a rotating disk is Euclidean. The report of Shu was not published in a scientific journal, but given to the library of Princeton University. His work is clearly that of an outsider. He came to his result by neglecting the reality of the Lorentz contraction as applied to measuring rods on a rotating disk, and concluded that the general theory of relativity is not correct.

W. Glaser [25] argued in 1934 that the line-element of flat spacetime in a rotating reference frame can not be separated in a temporal and a spatial part with no product terms between a time differential and a spatial differential, and concluded that space in a rotating frame is Euclidean. M. P. Langevin [26] reacted immediately and demonstrated explicitly that this is possible, by performing the following calculation.

Let marked co-ordinates refer to the inertial rest frame K' of the axis, and unmarked to co-moving frame K on the rotating disk. The transformation between these coordinate systems is

$$t' = t\,, \quad r' = r\,, \quad \theta' = \theta + \omega t\,, \quad z' = z \tag{5}$$

The spacetime line-element in the inertial frame has the form

$$ds^2 = -c^2\, dt'^2 + dr'^2 + r'^2\, d\theta'^2 + dz'^2 \tag{6}$$

The line-element in the rotating frame is

$$ds^2 = -\left(c^2 - r^2\omega^2\right)dt^2 + 2r^2\omega\, dt\, d\theta + dr^2 + r^2\, d\theta^2 + dz^2 \tag{7}$$

For $r < c/\omega$ this may be written as

$$ds^2 = -c^2\, d\tau^2 + d\sigma^2 \tag{8}$$

with

$$d\tau = \sqrt{1 - r^2\omega^2/c^2}\left(dt - \frac{r^2\omega}{c^2 - r^2\omega^2}\, d\theta\right) \tag{9}$$

and

$$d\sigma^2 = dr^2 + \frac{r^2\, d\theta^2}{1 - r^2\omega^2/c^2} + dz^2 \tag{10}$$

Here the spatial line-element $d\sigma$ represents the geometry on the space defined by $d\tau = 0$. Since $d\tau$ is not a perfect differential it cannot be integrated around a closed curve on the rotating disk. Hence the surface represented by $d\tau = 0$ has a discontinuity along a radial line (see Fig.5). As seen from eq.(10) it is this surface that has the spatial geometry discussed by Einstein, with the ratio between the circumference and the diameter being $\pi/\sqrt{1 - r^2\omega^2/c^2}$.

The meaning of the time interval $d\tau$ was made clear by Rosen [27]. He introduced a local inertial coordinate system momentarily at rest relative to the rotating system by the relations

$$dx = dr\,, \quad dy = Ad\theta\,, \quad d\tau = Bdt + Cd\theta \tag{11}$$

where A, B and C were determined so as to make

$$ds^2 = -c^2 d\tau^2 + dx^2 + dy^2 + dz^2 \tag{12}$$

This gives

$$A = \frac{r}{\sqrt{1 - r^2\omega^2 / c^2}}, \quad B = \sqrt{1 - r^2\omega^2 / c^2}, \quad C = -\frac{r^2\omega^2 / c^2}{\sqrt{1 - r^2\omega^2 / c^2}} \tag{13}$$

From this it follows that

$$d\sigma^2 \equiv dx^2 + dy^2 + dz^2 = dr^2 + \frac{r^2 d\theta^2}{1 - r^2\omega^2 / c^2} + dz^2 \tag{14}$$

and that the transformation (11) for τ is identical to eq.(9). Hence τ is the time measured in a local inertial system momentarily at rest relative to a reference particle in the rotating system, and $d\tau = 0$ represents simultaneity in local inertial frames comoving with the rotating frame and positioned for example along a circle about the axis. The clocks of the local inertial frames instantaneously at rest relative to points along such circles are Einstein synchronised so that the velocity of light is isotropic as measured with these clocks.

The impossibility of Einstein synchronising clocks around the circumference of a rotating disk has been perceived as a problem by F. Goy and F. Selleri [28]. They write:

> The existence of a synchronisation is physically strange because if the whole disk is initially at rest in the laboratory (inertial) frame K', with clocks near its rim synchronised with the regular procedure used for all clocks of K', then when the disk moves, accelerates, and attains a constant angular velocity, the clocks must slow their rates but cannot desynchronise for symmetry reasons, since they have at all times the same speed.

The clocks along the rim of the rotating disk are assumed to have at all times the same speed. This means that the acceleration program of each clock is identical as observed from K'. Assume that the angular acceleration is due to a succesion of blows at the rim. Then the blows are simultaneous in K'. Due to the relativity of simultaneity they are not simultaneous in the rest frame of an element on the rim. In such a frame the clocks at each end of the element get different velocities, which desynchronises the clocks.

Goy and Selleri are of course right in saying that the clocks remain synchronised in K'. Another way of obtaining an identical synchronisation to that which results from the procedure of Goy and Selleri, is to use a time signal emitted from the axis. This will reach all clocks at a circle with center at the axis simultaneously both as measured in K' and as measured in the rotating rest frame K of the disk. The clocks synchronised in this way are just the coordinate clocks on the rotating disk. They show the same time as the clock at the axis, as the clocks in K'. However, these clocks are not Einstein synchronised.

If one includes the postulate that the velocity of light is isotropic as part of the special theory of relativity, and demands that special relativity is valid locally, then local physical measurements must be performed by means of Einstein synchronised clocks. Globally, however, it may be advantageous to use coordinate clocks.

The 3-space on a rotating disk is defined to be everywhere orthogonal to the world lines of fixed particles on the disk. This means that this space is defined by simultaneity on the Einstein synchronised clocks. The space defined by simultaneity on the coordinate clocks, on the other hand, is the 3-space of the inertial rest frame of the axis, which is flat.

Reichenbach [29] discussed the spatial geometry on a rotating disk in 1924. He distinguished between what he called "the spatial geometry of the circular disk" (SGD) and "the geometry of rigid rods on the disk" (GRD). He defined SGD as the geometry found my measurements made at a fixed co-ordinate time $t =$ constant. Putting $dt = 0$ in eq.(7) he obtained for the line element of the surface $z =$ constant: $d\sigma_{t=const}^2 = dr^2 + r^2 d\theta^2$. Thus, he concluded that the spatial geometry of the rotating disk is Euclidean. However, the space defined by $t =$ constant is the simultaneity space of the inertial rest frame K' of the axis. It is not reasonable to identify the spatial geometry of the rotating disk with the geometry of this space. Reichenbach defined GRD by simultaneity in the local rest frame of a mass element on the disk. He then pointed out that this represents non-simultaneity in K', which cannot be defined globally in the rotating rest frame K of the disk. By considering rigid measuring rods on the disk he obtained the same line element as in eq.(14).

Also he gave a nice illustration of the difference between the two geometries. Imagine an arbitrary point P on the disk. Draw a circle around P with radius $d\sigma = 1$ (see Fig.1).

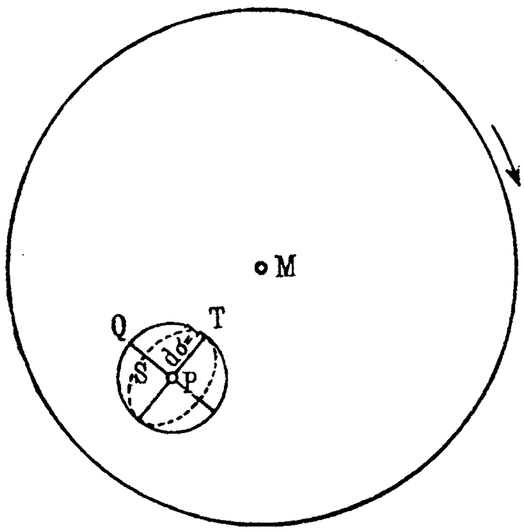


Fig.1. Rotating disk with measuring rod. The dashed ellipse is the curve followed by the end of the rod when it is rotated about P.

If a rigid rod of length 1 is laid through P in the radial direction, its characteristic length is $d\sigma_r = 1$. If the rod lies tangentially, and its length is measured as the distance between events at its ends, that are simultaneous in K', one finds a Lorentz contracted length for the measuring rod. If the rod is rotated about an axis through P, its end describes the dotted curve in Fig.14. This curve is an ellipse whose radial half axis = 1 and whose tangential half axis = $\sqrt{1-r^2\omega^2/c^2}$.

GRD means that the same rigid rod shall have the spatial length 1 independently of its position and orientation. In other words, the half axes of the dotted curve are called equally long. SGD, on the other hand, means that all the radii of the circle are equally long. Note that Fig.1 is drawn from the point of view of the non rotating frame K'.

In general an arbitrary spacetime line-element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{15}$$

may be separated in a temporal part $d\tau$ and a spatial part $d\sigma$ as in eq.(8) with

$$d\tau = \sqrt{-g_{tt}}\left(dt + \frac{g_{ti}}{g_{tt}} dx^i \right) \tag{16}$$

and

$$d\sigma^2 = \gamma_{ij}\, dx^i\, dx^j \,, \quad \gamma_{ij} = g_{ij} - \frac{g_{ti} g_{tj}}{g_{tt}} \tag{17}$$

J. W. Weysenhoff [30] used this separation and defined a local angular velocity angular velocity vector by

$$\vec{\omega} = \frac{1}{2}\frac{c}{\sqrt{-g_{tt}}}\left(\nabla \times \vec{\alpha} + \vec{\alpha} \times \frac{\partial \vec{\alpha}}{\partial t} \right), \quad \vec{\alpha} = \frac{g_{ti}}{g_{tt}} \vec{e}_i \tag{18}$$

He then showed that a condition for being able to Einstein synchronise clocks around a closed curve, or in other words, to introduce a coordinate system that is everywhere time orthogonal, is that $\vec{\omega} = 0$, and noted:

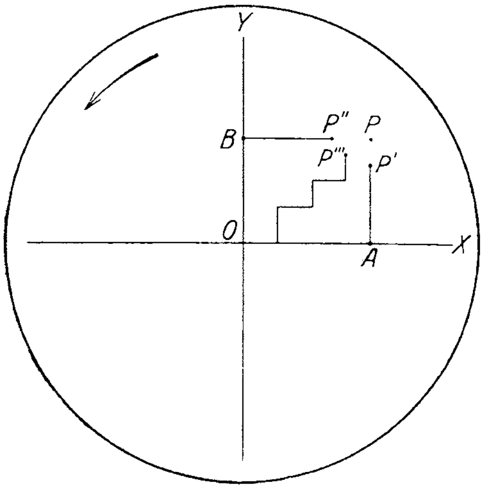
> We then see that for example on a rotating disk it is impossible everywhere to introduce a time orthogonal coordinate system. Expressed in another way: It is impossible to synchronise the clocks everywhere on the disk so that all light signals are symmetrical, that is, so that the velocity of light is the same in every direction.

In passing we note that this explains the result of Sagnac's experiment [31] from the point of view of observers on the rotating disk [32].

Einstein also noted that the properties of measuring rods and clocks on a rotating disk illustrates a general fact [16]:

> In the general theory of relativity space and time cannot be defined in such a way that differences of the spatial co-ordinates can be directly measured by the unit measuring rod, or differences in time co-ordinate by a standard clock.

This was illustrated very explicitly in connection with the rotating disk by H. Thirring, who also noted that there are tangential stresses in the material of a rotating disk, writing [33]:



Man denke sich eine ebene Kreisscheibe in Rotation gegen ein Inertialsystem versetzt. Die radien *r* der Scheibe werden von der Lorentzkontraktion nicht betroffen, weil sie senkrecht zur Bewegungsrichtung stehen. Die Peripherie wird in ihrem Bestreben, sich zuzammenzuziehen, durch die Kohäsionskräfte gehindert; die Lorentzkontraktion wird durch die Dehnung kompensiert, die von den elastischen spannungen der in sich zusammenhängenden Peripherie bewirkt wird. Dagegen erleidet ein längs der Scheibenumfangs angelegter spannungsfreier Masstab die Lorentzkontraktion, was zur Folge hat, dass eine Ausmessung des Verhältnisses zwischen Umfang und Peripherie einen höheren Wert als 2π liefern muss. Es treten also Abweichungen von den Gesetzen der euklidischen Geometrie auf, in analoger Weise, wie sie sich bei entsprechenden Messungen auf gekrümmten Flächen ergeben mussen. Zeichnet man etwa auf der Erdoberfläche einen Kreis und misst man das Verhältnis zwischen Kreisumfang und dem längs der Erdoberfläche selbst gemessenen Radius, so ergibt sich ebenfalls ein von 2π abweichender Wert und zwar, der positiven Krümmung der Kugel entsprechend, ein kleineren Wert als 2π, während in dem obenerwähnten Beispiel der rotierende Scheibe das Verhältnis grösser als 2π wird, was einer "negativen Krümmung" entspricht. Es zeigt sich ferner, dass die übliche Art der Koordinatenbestimmung von Punktereignissen mit hilfe von Uhrenangaben und kartesischen Koordinaten nicht mehr zu eindeutigen Ergebnissen führt, wie aus dem nachstehenden einfachen Beispiel hervorgeht: Es werde auf der rotierenden Scheibe ein rechtwinkliges Koordinatensystem *XY* gezeichnet, dessen uhrsprung mit dem Sheibenmittelpunkt zuzammenfällt, und es sei die Aufgabe gestellt, den punkt *P* mit den Koordinaten *x,y* zu finden. Das kan man nun zunächst so machen , dass man einen Einheitmasstab längs der *X*-Achse *x*mal aufträgt, dadurch gelangt man in den Punkt *A*; dort errichtet man eine Senkrechte, längs derer man den Einheitsmasstab *y* mal auträgt. Bei dieser letzteren Operation ist aber der Masstab gemäss dem oben Gesagten verkürzt; man gelangt also nicht in jenen Punkt *P*, der auf der ruhenden Scheibe die Koordinaten *x* und *y* hätte, sondern in einer näher an *A* gelegenen Punkt *P'*. – Würde man dagegen in den punkt *P* gelangen wollen, indem man mit dem Auftragen eines Einheitsmasstabes längs der *Y*-Achse beginnt, so würde man aus dem gleichen Grunde in einen Punkt *P"* gelangen, der näher zur *Y*-Achse liegt. Wenn man ferner einen dritten Weg, z.B. den in der Figur gezeichneten Treppenweg ginge, so würde man noch in einen anderen punkt *P"'* kommen usw..

A strange objection to Einstein's analysis of the spatial geometry in a rotating frame was given by Atwater [34]. He argued against the special relativistic assumption that the measuring rods along the circumference of a rotating disk are Lorentz contracted as observed from the inertial rest frame K' of the axis, writing:

> Consider a circular disk, made possibly of transparent material, on the circumference of which a pattern of stripes is painted in four octants, as indicated in Fig.2a.

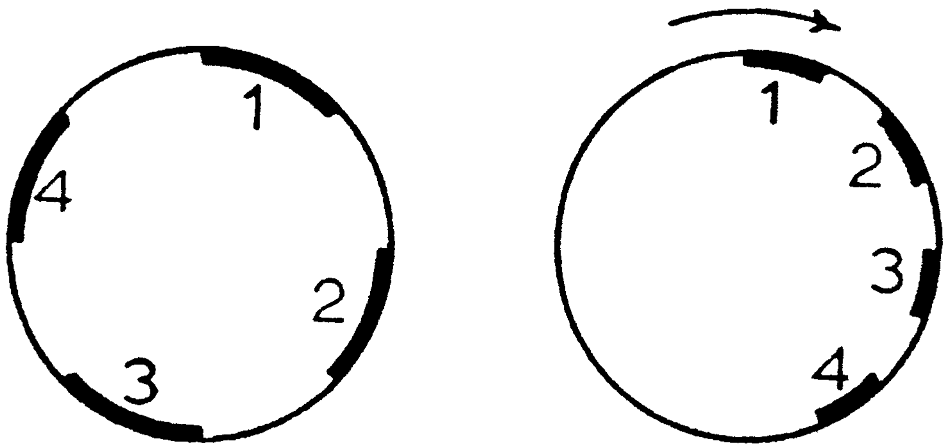


Fig.2a. Stationary disk. **2b.** Result of an arbitrary application of Lorentz contraction to periphery of a disk in rotation.

> The disk is then set into rotation at a speed such that the rim is travelling at 86,6 per cent of the velocity of light in free space, for a Lorentz contraction factor of ½. The positions of the ends of the stripes on the rim can be observed at an instant of laboratory time. This is possible by means of a flash of light emitted above the plane of the disk which exposes the shadows of the stripes on a photographic film held stationary in the back of the disk; an alternative coincidence-detection observation may also be devised. An elementary application of the special relativistic hypothesis could then lead to the expectation of a pattern as in Fig. 2b, which is clearly impossible on the basis of the symmetry of the disk. The special relativistic assumption must therefore be discarded.

The argument of Atwater is not valid, however, because painted marks on the circumference of the disk will not appear Lorentz contracted unless no tensions appear on the disk, i.e. unless it is put into rotation in a Born rigid way. As will be shown below, this is not possible. Furthermore, as noted by Suzuki [35] the paradoxical, non-symmetric situation of Fig. 2b is not predicted by the theory of relativity. If there were a contraction, the circumference of the disk would contract uniformly, and no asymmetry would result. Further replies to Atwater's letter are found in ref. [36].

Spatial geodesics on the rotating disk

C. Møller [37] and H. Arzeliès [38] have made some interesting observations concerning the spatial geometry of a rotating reference frame. For one thing they gave a nice illustration of the non-Euclidean character of this geometry by calculating spatial geodesics on the surface $z = \text{constant}$, $\hat{t} = \text{constant}$.

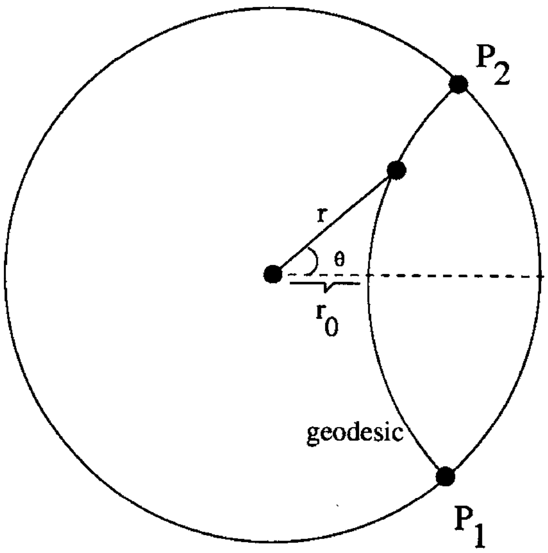


Fig. 3. Spatial geodesic curve on a rotating disk.

We consider a geodesic curve between two points P_1 and P_2 on the periphery of the disk, as shown in fig.2. The Lagrangian function of the curve is

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{r^2\dot{\theta}^2}{1 - r^2\omega^2/c^2} \tag{19}$$

where $u^\mu = \dot{x}^\mu$ are the components of the unit tangent vector field of the curve. Hence $\mathbf{u}\cdot\mathbf{u} = 1$, which gives

$$\dot{r}^2 + \frac{r^2\dot{\theta}^2}{1 - r^2\omega^2/c^2} = 1 \tag{20}$$

Since θ is a cyclic coordinate

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = \frac{r^2\dot{\theta}}{1 - r^2\omega^2/c^2} = constant \tag{21}$$

or

$$\dot{\theta} = \left(1 - \frac{r^2\omega^2}{c^2}\right)\frac{p_\theta}{r^2} \tag{22}$$

Inserting eq.(22) into eq.(20) gives

$$\dot{r}^2 = 1 - \left(1 - \frac{r^2\omega^2}{c^2}\right)\frac{p_\theta^2}{r^2} \tag{23}$$

This leads to the equation of the geodesic curve between P_1 and P_2

$$\frac{dr}{d\theta} = \pm\frac{r^2\sqrt{1 - \left(1 - \frac{r^2\omega^2}{c^2}\right)\frac{p_\theta}{r^2}}}{\left(1 - \frac{r^2\omega^2}{c^2}\right)p_\theta} \tag{24}$$

Inserting the boundary conditions $\dot{r} = 0$, $r = r_0$ for $\theta = 0$ into eq.(23) we get

$$\frac{p_\theta}{r_0} = \sqrt{1 + \frac{p_\theta^2 \omega^2}{c^2}} \tag{25}$$

Using this one can rearrange eq.(24) obtaining

$$\frac{dr}{r\sqrt{r^2 - r_0^2}} - \frac{\omega^2}{c^2} \frac{r\,dr}{\sqrt{r^2 - r_0^2}} = \frac{d\theta}{r_0} \tag{26}$$

Integration now yields

$$\theta = \pm \frac{r_0 \omega^2}{c^2} \sqrt{r^2 - r_0^2} \mp arc\cos\frac{r_0}{r} \tag{27}$$

One such curve is plotted in fig.3. We see that the geodesic is curved inwards when the disk is rotating. This is intuitively reasonable since the tangential measuring rods in the rotating frame are longer, as observed from the non-rotating laboratory frame, the farther inwards they are. Hence, to have fewer measuring rods along the curve it should pass closer to the axis. On the other hand this bending makes the curve longer as observed in the inertial rest frame of the axis. The shape of the curve is represents a compromise between these two opposing effects.

Further properties of spacetime geodesics for material particles and photons and of spatial geodesics in a rotating reference frame have been discussed by Ashworth and Davies [39].

Relativity of simultaneity and coordinates in rotating frames

P. Franklin [40] argued in 1922 that the "Galilean" transformation (5) is not suitable in the context of the special theory of relativity. He suggested that one should apply a Lorentz like transformation

$$t = \frac{t' - v(r')\,r'\theta'/c^2}{\sqrt{1 - v(r')^2/c^2}}, \quad r = r', \quad r\theta = \frac{r'\theta' - v(r')\,t'}{\sqrt{1 - v(r')^2/c^2}} \tag{28}$$

and calculated the velocity $v(r)$ as follows. Eq.(28) leads to the relativistic formula for velocity addition in the form

$$\omega\,\Delta r' = \frac{v(r' + \Delta r') - v(r')}{1 - v(r' + \Delta r')\,v(r')/c^2} \tag{29}$$

where ω is a constant angular velocity. Taking the limit $\Delta r' \to 0$ leads to

$$\omega \frac{dr'}{dv} = \frac{1}{1 - v^2/c^2} \tag{30}$$

Integrating with $v(0) = 0$ gives

$$v = c \tanh \frac{r'\omega}{c} \tag{31}$$

which is less than c for every finite value of r' and ω. Inserting this into eq.(28) finally leads to the transformation formula

$$t = t' \cosh \frac{r'\omega}{c} - \frac{r'\theta}{c} \sinh \frac{r'\omega}{c}, \quad r = r', \quad \theta = \theta' \cosh \frac{r'\omega}{c} - \frac{ct'}{r'} \sinh \frac{r'\omega}{c} \tag{32}$$

The transformation (32) has later been discussed by Trocheris [41] and Takeno [42]. Trocheris noted that the coordinate clocks in the rotating system are Einstein synchronised. Hence, the coordinate velocity of light is isotropic in this system.

The rotating coordinate system obtained by this transformation has, however, some disadvantages compared to that obtained by the transformation (5). If one calculates the spatial line element of the space defined by putting $t = \text{constant}$ in eq.(32), one obtains a time dependent spatial metric in spite of the fact that the system rotates with constant angular velocity. Also, the simultaneity, say $t = 0$, of the rotating coordinate system corresponds to

$$t' = \frac{r'\theta'}{c} \tanh \frac{r'\omega}{c} \tag{33}$$

in the inertial rest frame of the axis of rotation. Hence, going around a circle about the axis of rotation one arrives at a different point of time than at the start. This means that a certain event corresponds to different points of time in the rotating coordinate system. In other words there exists a time discontinuity along a radial line in this coordinate system.

L. Herrara [43] has recently discussed the above transformation and presented a modified form of it. A thorough discussion of the co-ordinate system above, and several other transformations to rotating frames, have been given by B. Chakraborty and S. Sarkar [44].

It should be noted that one may always introduce an orthonormal basis field with Minkowski metric at arbitrary points of a rotating disk. T. A. Weber [45] has explicitly demonstrated how this can be done, by giving the Lorentz transformation from the inertial rest frame K' of the axis to such a basis. As he pointed out, such a transformation has only a local geometrical significance, and must be given in terms of differentials that are not exact. The transformation is not integrable. This was emphasised also by J. F. Corum [46]. H. Nicolić [47] has recently described the relativistic kinematics with reference to a field of local Fermi frames comoving with a rotating disk.

A recent preprint by V. Bashkov and M. Malakhaltsev [48] contains a misunderstanding that should be clarified. They make a “Lorentz transformation” in differential form to “infinitesimal coordinates” on the rotating disk,

$$d\hat{r} = dr' \ , \quad \hat{r}\, d\hat{\theta} = \frac{r' d\theta' - r'\omega\, dt'}{\sqrt{1-\frac{r'^2\omega^2}{c^2}}} \ , \quad d\hat{t} = \frac{dt' - \frac{r'^2\omega}{c^2} d\theta'}{\sqrt{1-\frac{r'^2\omega^2}{c^2}}} \tag{34}$$

This transformation preserves the form of the line element. Hence,

$$ds^2 = -c^2 d\hat{t}^2 + d\hat{r}^2 + \hat{r}^2 d\hat{\theta}^2 \tag{35}$$

Bashkov and Malakhaltsev note that the transformation is not integrable and points out that it is therefore impossible to express $\hat{t}, \hat{r}, \hat{\theta}$ as functions of t', r', θ' in a finite way.

The clocks measuring the coordinate time $\hat{t}$ are Einstein synchronized. Hence the geometry on the rotating disk is defined by the simultaneity $d\hat{t} = 0$. Then putting $d\hat{t} = 0$ in eq.(35) one obtains the spatial line element

$$dl^2 = d\hat{r}^2 + \hat{r}^2 d\hat{\theta}^2 \tag{36}$$

From this Bashkov and Malakhaltsev concludes:

> Thus on the disk we get the Euclidean geometry, contrary to the conclusions of other researchers who obtained the spatial line element (10).

However, due to the local character of the non-coordinate basis field introduces by Bashkov and Malakhaltsev one cannot deduce the geometry of space just by inspecting the form of the line element. The curvature of the space must be calculated from the general formulae including the structure coefficients [49].

The comoving orthonormal basis of the rotating disk corresponding to the transformation (34) has spatial basis forms

$$\omega^{\hat{r}} = dr' \ , \quad \omega^{\hat{\theta}} = \frac{r' d\theta' - r' \omega \, dt'}{\sqrt{1 - r'^2 \omega^2 / c^2}} \tag{37}$$

Using GRTensor to calculate the Ricci curvature scalar $R = R_{\hat{i}\hat{j}} R^{\hat{i}\hat{j}}$ and the Kretschmann curvature scalar $K = R_{\hat{i}\hat{j}\hat{m}\hat{n}} R^{\hat{i}\hat{j}\hat{m}\hat{n}}$ one obtains

$$R = -\frac{6\omega^2}{\left(1 - r'^2 \omega^2 / c^2\right)^2} \ , \quad K = \frac{36\omega^4}{\left(1 - r'^2 \omega^2 / c^2\right)^4} \tag{38}$$

showing that the simultaneity space $d\hat{t} = 0$ is curved.

What is the effect of the Lorentz contraction upon a disk that is put into rotation?

Having seen Einstein's explanation making it clear that one measures a longer circumference on a rotating disk the faster it rotates, let us go back once more to 1910 and see a consequence of taking seriously the supposition of Ehrenfest's paradox that the circumference of the rotating disk itself is Lorentz contracted. In an article published in 1910 G. Stead and H. Donaldson [50] gave an analysis of the geometrical properties of a rotating disk based upon the apprehension that the periphery of a rotating disk is contracted and the radius not. They treated the disk as an elastic membrane able to deform without any resistance. Hence the Lorentz contraction in the direction of motion of each element of the membrane forces it to bend, so that it gets the shape of a cup as shown in figure 4.

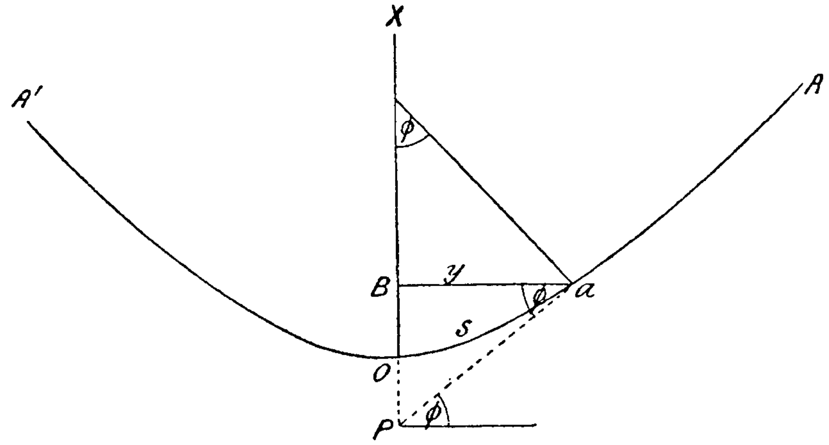


Figure 4. The shape of a relativistically rotating disk bent due to the Lorentz contraction in the tangential direction.

With reference to figure 1 their analysis is as follows: If $A'OA$ represents the vertical section of the final form of the disk with axis of rotation OX, then Oa measured along the arc is equal to r, while aB measured perpendicular to OB will be $r\sqrt{1-v^2/c^2}$. Writing $Oa = s$ and $aB = y$ we have

$$y = s\sqrt{1-v^2/c^2} = s\sqrt{1-y^2\omega^2/c^2} \tag{39}$$

ω being the angular velocity of the disk. Hence

$$y = \frac{s}{\sqrt{1+s^2\omega^2/c^2}} \tag{40}$$

which leads to

$$\cos\phi \equiv \frac{dy}{ds} = \frac{1}{\left(1+s^2\omega^2/c^2\right)^{3/2}} \tag{41}$$

The velocity of an arbitrary point on the disk due to the rotation is

$$v = y\omega = \frac{c}{\sqrt{1 + c^2 / s^2\omega^2}} \qquad (42)$$

The velocity is less than c for all values of y and ω. Equation (41) shows that when ω becomes very large, $\cos\phi$ is small and nearly independent of s. Hence, for large ω the former disk approaches the form of a right circular cone of small angle, except near the centre where it is curved. A similar description was given by M. Galli [52]

Stead and Donaldson also noted that if the disk were forced not to bend during the rotation, then its material would be strained.

In his popular introduction to the theory of relativity [53] A. S. Eddington comments on Ehrenfest's paradox. He considers a rapidly revolving wheel and writes:

> Each portion of the circumference is moving in the direction of its length, and might be expected to undergo the Fitzgerald contraction due to its velocity; each portion of a radius is moving transversely and would therefore have no longitudinal contraction. It looks as though the rim of the wheel should contract and the spokes remain the same length, when the wheel is set revolving. The conclusion is absurd, for a revolving wheel has no tendency to buckle – which would be the only way of reconciling these conditions. The point which the argument has overlooked is that the results here appealed to apply to unconstrained bodies, which have no acceleration relative to natural tracks in space. Each portion of the rim of the wheel has a radial acceleration, and this affects its extensional properties. When acceleration as well as velocities occur a more far-reaching theory is needed to determine the changes of length.

Comparing this statement with Einstein's considerations we see that they are concerned with different problems. Like Planck, Eddington talks about what happens to a disk that is set into rotation. Einstein, on the contrary, is not interested in this. He considers a disk rotating with constant angular velocity and does not compare of the disk when it is a t rest and when it rotates. He is interested in the results of measurements of the length of the circumference and a radial line as performed by means of standard measuring rods corotating with the disk. This is a purely kinematical problem, while Planck and Eddington consider a dynamical problem involving the relativistic theory of elastic media.

This latter problem was treated by H. A. Lorentz [54] and Eddington [55] in 1921 and 1923, respectively. Without giving details Lorentz reports that he has applied the mentioned theory to a thin disk and found a relativistic deformation:

> The result is that, if v is the velocity at the rim, the radius will be shortened in the ratio of 1 to $1-(1/8)(v^2/c^2)$. The circumference changing to the same extent, its decrease is seen to

> be exactly one fourth of that of a rod moving with the same velocity in the direction of its length.

This is the same result as one of Ignatowsky's [12].

Lorentz adds a comment which seems to indicate that the principle of relativity is not valid for rotating motion:

> At first sight our problem seems to lead to a paradox. Let there be two equal disks A and B, mounted on the same axis, A revolving and B at rest. Then A will be smaller than B, and it must certainly appear so (the disks being assumed to be quite near each other) to any observer, whatever be the system of coordinates he chooses to use. However, we can introduce a system of coordinates K revolving with the disk A; with respect to these it will be B that rotates, and so one might think that now this latter disk will be the smaller of the two. The conclusion would be wrong because the system *K* would not be a normal one. If we leave S for it, we must at the same time change the potentials g_{ab}, and if this is done the fundamental equations will certainly again lead to the result that A is smaller than B.

A similar paradox of an electromagnetic nature was resolved by L. I. Schiff [56] in 1939, indicating how the principle of relativity may be extended to encompass rotating motion in the general theory of relativity. (A thorough discussion of this question is found in [57].)

Eddington arrived at the same result for the deformation as Lorentz. Let us follow his deduction. A disk made of homogeneous, incompressible material is caused to rotate with angular velocity ω. Eddington wants to calculate the alteration in radius due to the Lorentz contraction of its mass elements. The meaning of *incompressible* is that the particle density σ (referred to proper measure) is constant and equal to that of a non-rotating disk. But for a rotating disk the particle density σ' referred to axes fixed in space is different.

Since the number of particles in a comoving volume is invariant, then due to the Lorentz contraction,

$$\sigma' = \sigma\left(1 - r^2\omega^2 / c^2\right)^{-1/2} \tag{43}$$

since $r\omega$ is the velocity of a mass element at a distance r from the axis of rotation. The proper volume element of the disk is

$$dV = \left(1 - r'^2\,\omega^2 / c^2\right)^{-1/2} r'dr'd\theta'dz' \tag{44}$$

If the thickness of the disk is b, and its boundary is given by $r' = a'$, the total number of particles in the disk will be

$$N = 2\pi\sigma b \int_0^{a'} \left(1 - r^2\omega^2 / c^2\right)^{-1/2} r'dr' \tag{45}$$

Since this number is unaltered by the rotation, a' must be a function of ω such that

$$\int_0^{a'} \left(1 - r^2\omega^2 / c^2\right)^{-1/2} r' dr' = \text{constant} \tag{46}$$

or

$$\frac{1}{\omega}\left(1 - \sqrt{1 - a'^2 \omega^2 / c^2}\right) = \text{constant} \tag{47}$$

Expanding the square root this gives approximately

$$\frac{1}{2} a'^2 \left(1 + \frac{1}{4} a'^2 \omega^2 / c^2\right) = \text{constant} \tag{48}$$

so that if a is the radius of the disk at rest

$$a'\left(1 + \frac{1}{8} a'^2 \omega^2 / c^2\right) = a \tag{49}$$

Hence to the same approximation

$$a' = a\left(1 - \frac{1}{8} a'^2 \omega^2 / c^2\right) \tag{50}$$

Here a' is the radius of the disk referred to its corotating rest frame, but a transformation to the stationary rest frame in which the axis is at rest does not change the radius. Hence, a' is equal to the radius of the rotating disk as measured in the inertial rest frame of its axis.

F. Winterberg [58] has recently deduced the result of Lorentz and Eddington from the theory of elastic media, and also generalized their result by taking account of different stresses in the radial and the tangential directions.

Curved space and discussion of Einstein's and Eddington's analyses of the rotating disk

In 1942 Berenda published a discussion of the spatial geometry of the surface of a rotating disk [59]. He came with some interesting observations, but also made a few misinterpretations of earlier works. Under the heading "Einstein's Geometry" he correctly noted that a standard measuring rod along the periphery of a rotating disk is Lorentz contracted. The inertial rest frame of the axis of the disk is called *G*. Berenda then wrote:

> If we add up the measurement results of the G observers all around the disk, we find
>
> $$\overline{C} = \sum \overline{P} = \sum P'\left(1-\frac{v^2}{c^2}\right)^{1/2} = C'\left(1-\frac{v^2}{c^2}\right)^{1/2} < C' \qquad (51)$$
>
> where $\sum P' = C' = 2\pi r$ is the circumference of the disk when it is at rest in the *G* frame, and $\overline{C}$ is the circumference of the rotating disk relative to the *G* observers. Then
>
> $$\overline{C} = 2\pi r\left(1-\frac{v^2}{c^2}\right)^{1/2} < 2\pi r \qquad (52)$$

However, according to the Einstein citations above $\overline{C} > 2\pi r$. Einstein's point is that because each measuring rod along the periphery is Lorentz contracted there is place for more of them around the circumference the faster the disk rotates, and the length of the circumference is just the number of measuring rods around it.

Then, under the heading "Eddington's Geometry", Berenda claimed that there is an error in Eddington's treatment cited above. He noted that in the transition from eq.(44) to eq.(45) Eddington assumed that the angle around the periphery of the rotating disk is 2π just as for a disk at rest. Then he wrote:

> This is, however, the whole point at issue, since the latter assumption is equivalent to the postulate that angular measures are unaffected by rotation, i.e., that the geometry remains Euclidean.

However, Eddingtons's result is correct because the angle around a circle with an arbitrary centre on a curved surface is defined on the tangent plane of the centre of the circle, i.e. it is defined locally. Hence, in the case of a circle around the axis of a rotating disk one has to take the limit $r \to 0$ to find the angle. This implies that the angle around a circle *is* 2π even on a surface with non Euclidean geometry.

Berenda then deduced eq.(10) above and pointed out that this line element gives the geometry of a surface that is everywhere orthogonal to the world lines of points fixed on the disk. In the case of the rotating disk the surface is curved and described by the line element

(10). He then found the non vanishing components of the Riemann curvature tensor for this surface

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112} = -\frac{3r^2\omega^2}{c^2}\left(1-\frac{r^2\omega^2}{c^2}\right)^{-3} \tag{53}$$

The Gaussian curvature of the surface is

$$K = \frac{R_{1212}}{g} = -\frac{3\omega^2}{c^2}\left(1-\frac{r^2\omega^2}{c^2}\right)^{-2} \tag{54}$$

where $g = \det\left(g_{ij}\right)$. Hence the surface orthogonal to the world lines of particles fixed on the rotating disk has negative curvature. The geometry is hyperbolic.

In a recent work Rizzi and Ruggiero [4] have given a thorough analysis of the space geometry of rotating platforms. The authors have applied Catteano's projection formalism to define precisely the concept of spatial geometry in a rotating frame in an operationally meaningful way. Their rigorous approach has reproduced what may be called the standard results leading to a hyperbolic spatial geometry with essentially the curvature given above.

Uniform contra rigid rotation

Hill [60] argued in 1946 that the limitation $r < c/\omega$ of the extension of a rotating disk coming from the Galilean form of the transformation (5) combined with the relativistic requirement that the velocity of each point of the rotating frame must be less than the velocity of light, seems unnatural. He therefore proposed to specify the motion of the disk so that close to the axis the rotational velocity is approximately proportional to the radius i.e. the angular velocity is approximately constant, and far from the axis the velocity approaches that of light. In addition he demanded that the disk consists of a fluid rotating uniformly, according to the following definition:

> The motion of the fluid will be considered to be uniform rotation with constant angular velocity ω_0 if the relative angular velocity of the material about any point *P*, as measured in a set of axes with respect to which *P* is momentarily at rest, has the value ω_0 independently of the choice of the point *P*.

The set of axes G' with respect to which P is momentarily at rest is defined by a Lorentz transformation from the rest frame of the axis, i.e. it is the comoving inertial rest frame of the point P. The angular velocity is defined in G' by

$$\vec{\omega}' = \frac{1}{2}\nabla \times \vec{v}' \tag{55}$$

where $\vec{v}'$ is the velocity field of the disk material with respect to G'. The velocity of a point at a distance r from the axis with respect to the inertial rest frame G of the axis, has components

$$v_x = -y\,\omega(r),\;\; v_y = x\,\omega(r) \;\; with \;\; r = \left(x^2 + y^2\right)^{1/2} \tag{56}$$

Hence

$$v(r) = r\,\omega(r) \tag{57}$$

Calculating $\vec{v}'$ from eq.(57) and the Lorentz transformation between G and G' and $\vec{\omega}'$ from eq.(55) and then demanding that $\vec{\omega}' =$ constant, Hill obtained a differential equation for $v(r)$. Solving this he found an expression involving Bessel functions, a result which was later found also by Prechtl [61]. In the limiting cases of large and small distances from the axis of rotation the expression of Hill gives

$$v(r) = r\omega'\left(1 - \frac{1}{2}\frac{r^2\omega'^2}{c^2}\right) \;\; for \;\; \frac{r\omega'}{c} << 1 \tag{58}$$

$$v(r) = c\left(1 - \frac{1}{4}\frac{c}{r\omega'} - \frac{1}{32}\frac{c^2}{r^2\omega'^2} - \cdots\right) \;\; for \;\; \frac{r\omega'}{c} >> 1 \tag{59}$$

As pointed out by Hill himself his discussion did not solve the problem posed in Ehrenfest's paradox. This would demand a theory of the generation of motion from a state of rest under suitable specifications.

Hill's discussion was followed up by N. Rosen [62]. He criticised Hill for making use of a Lorentz transformation in going over to a non inertial frame of reference. However, Hill only used the Lorentz transformation to go into an instantaneous rest frame of a point on the rotating disk, which is clearly a valid procedure.

Relativistically rigid motion and rotation

In a subsequent article [27] Rosen applied the concept of relativistic rigid motion as introduced by Born [5] to the case of uniform rotation in order to deduce the function $v(r)$ for this case. He argued as follows:

> In classical physics one can characterize the motion of a rigid body by the fact that the rate of strain vanishes. In a Cartesian coordinate system, the velocity components satisfy the relation
>
> $$\partial v_i / \partial x_k + \partial v_k / \partial x_i = 0 \quad \left(i, k = 1, 2, 3\right) \tag{60}$$

In a relativistic treatment one looks for a covariant equation which reduces to (55) in a Galilean system if the velocity is small (compared to that of light). The obvious generalization is to introduce, in an arbitrary coordinate system, the symmetrical tensor

$$P_{\mu\nu} = \frac{1}{2}\left(u_{\mu;\nu} + u_{\nu;\mu}\right) \tag{61}$$

where the velocity 4-vector u^λ is given by $u^\lambda = dx^\lambda / ds$ and a semicolon denotes covariant derivation, and take as the condition for rigid-body motion $P_{\mu\nu} = 0$. However, this condition represents too severe a restriction. In view of the fact that the vector u^λ satisfies the identity $u^\lambda u_\lambda = -1$ it follows from $P_{\mu\nu} = 0$ on multiplication by u^ν that $u_{\mu;\nu}u^\nu = 0$. This is just the covariant form of the condition that the four acceleration vector vanishes, so that every particle of the body moves along a geodesic, or in a gravitation free space, along a straight line.

It is therefore necessary to weaken the condition imposed on the motion. For this purpose we replace $P_{\mu\nu}$ by

$$p_{\mu\nu} = \frac{1}{2}\left(u_{\mu;\nu} + u_{\nu;\mu} - u_{\mu;\alpha}u^\alpha u_\nu - u_{\nu;\alpha}u^\alpha u_\mu\right) \tag{62}$$

since we then have the identity $p_{\mu\nu}u^\nu \equiv 0$. Let us now take as the condition for rigid-body motion

$$p_{\mu\nu} = 0 \tag{63}$$

In a Galilean coordinate system, at a point where the velocity 3-vector vanishes, this reduces to the classical condition (60). Therefore it is equivalent to the condition proposed by Born [5].

Since eq.(58) is a tensor equation it can be applied in any frame of reference. If we take an inertial system with Cartesian coordinates then we can describe rotation about an axis by setting

$$u_1 = u^1 = \sigma y, \quad u_2 = u^2 = \sigma x, \quad u_3 = u^3 = 0, \quad u_4 = -u^4 = -\left(1 + \sigma^2 r^2 / c^2\right)^{1/2} \tag{64}$$

with $\sigma = \sigma(r), \quad r^2 = x^2 + y^2$. From eq.(63) one gets the single equation

$$d\sigma / dr = \sigma^3 r \tag{65}$$

which has for its solution

$$\sigma = \omega / \left(1 - r^2\omega^2 / c^2\right)^{1/2} \tag{66}$$

where ω is a constant. Going over to the three-dimensional velocity v by the ralations

$$u = \left(u_1^2 + u_2^2\right)^{1/2} = \sigma r = v / \left(1 - v^2 / c^2\right)^{1/2} \tag{67}$$

we get

$$v = r\omega \tag{68}$$

Hill [60] defined uniform rotation as a motion where the local angular velocity of the disk material as measured in local inertial frames in which the material instantaneously has no translational motion, is independent of the position. The calculation of Rosen [27] shows that uniform rotation in this sense is not Born rigid.

The theory of elastic media applied to the rotating disk

The results of Lorentz [54] and Eddington [55] were reviewed and carried further by G. L. Clark [63, 64]. According to the non-relativistic theory of elastic media an elastic disk will get an increase of its radius when it is put into rotation. For an approximately incompressible disk ($\lambda >> \mu$, where λ and μ are the Lame constants of isotropic stress and shear, respectively) with radius a the radial displacement is

$$\Delta r_N = \frac{a^3 \omega^2}{8c_0^2} \tag{69}$$

where c_0 is the velocity of sound in the medium.

Taking into account the elastic effect of the tendency of the material to Lorentz contract in the tangential direction, there appears a tangential stress, which forces the disk to contract in the radial direction by an amount

$$\Delta r_R = -\frac{a^3 \omega^2}{8c^2} \tag{70}$$

Hence, the change of radius of the disk is

$$\Delta r = \frac{a^3 \omega^2}{8} \left(\frac{1}{c_0^2} - \frac{1}{c^2} \right) \tag{71}$$

By taking c_0 to be infinite for an incompressible medium one obtains the result of Lorentz and Eddington as given in eq.(50).

Clark notes, however, that according to the special theory of relativity the upper limit for the sound velocity is the velocity of light. Hence, relativistically a medium in which $c_0 = c$ is maximally rigid. For a disk consisting of such material $\Delta r = 0$. In this case the contraction of the disk due to the stresses induced by the efforts of the material to try to Lorentz contract is cancelled by the "classical elastic expansion" of the rotating disk.

Clark's analysis was followed up by Cavalleri [65] in 1968 in an interesting, although somewhat controversial paper. He first gave a thorough review of earlier work on this topic. Then he concluded that "Ehrenfest's paradox cannot be resolved from a purely kinematical point of view", and inferred that the relativistic kinematics for extended bodies is not generally self-consistent. Finally he noted that the analysis of Clark [63, 64] is valid only for small strains, and gave a more general analysis for material in which the sound velocity is equal to the velocity of light. While Clark found that the radius of such a disk is independent of the angular velocity, Cavalleri found that it increases with the angular velocity.

A few months later A. Brotas [66] followed up by calculating an explicite expression for the radius of a rotating ring consisting of the type of material considered by Cavalleri. McCrea [67] made the same calculation a few years later. Let us follow the main points of their calculation. Consider an element of the ring with proper length L_0^0 when the material is unstrained and proper length L_0 when it is strained. Let ρ_0^0 be the proper density of the unstrained material. Then the tension p is given by

$$p = \frac{1}{2}\rho_0^0 c^2 \left(\frac{1}{s^2} - 1\right), \quad s = \frac{L_0}{L_0^0} \tag{72}$$

The density of the strained material is

$$\rho_0 = \frac{1}{2}\rho_0^0 \left(\frac{1}{s^2} + 1\right) \tag{73}$$

The radius and length of the ring fulfils the Euclidean relationship $L = 2\pi R$ whether it is strained or not. Hence the Lorentz contraction of the ring when it rotates will cause a decrease of its radius. Let $2\pi R$ be the Lorentz contracted length or the rotating ring, which is strained due to the centrifugal effect of the rotation, i.e. it is the length of the ring as measured in the inertial rest frame of the axis. Then its proper length is $\gamma 2\pi R$, where $\gamma = \left(1 - R^2\omega^2 / c^2\right)^{-1/2}$. Thus,

$$s = \gamma \frac{R}{R_0} \tag{74}$$

where $2\pi R_0$ is the proper length of the ring when it is at rest so that it is not strained. Cavalleri showed that the tension of the material in the rotating ring is given by

$$p = -\rho_0 R^2 \omega^2 \tag{75}$$

From eqs.(72), (73) and (74) follow

$$s = \left(\frac{1 + R^2\omega^2 / c^2}{1 - R^2\omega^2 / c^2}\right)^{1/2} \tag{76}$$

Eqs.(74) and (76) finally lead to

$$R = \frac{R_0}{\sqrt{1 - R_0^2\omega^2 / c^2}} \tag{77}$$

This expression shows how the ring is elongated with increasing angular velocity. The maximal radius is obtained when $R\omega = c$. Then $s = \infty,\ \rho_0 = (1/2)\rho_0^0 \quad p = -(1/2)\rho_0^0 c^2 = -\rho_0 c^2,\ R = R_0\sqrt{2}$. It may be noted that the equation of state is that of vacuum energy. R.G. Newburgh [68] presented a practical proposal for the experimental investigation of the type of motion studied by Cavalleri, Brotas and McCrea. It may also be mentioned that the results of applying radar measurements and triangulation to a rotating disk have been analysed by R. C. Jennison, D. G. Ashworth and P. A. Davies [69-74]. Some of the results in these articles were earlier found by Arzeliès [38] in a comprehensible treatment of spacetime geodesics and spatial geodesics on a rotating disk. Furthermore, photographing of a rotating disk has been discussed by P. F. Browne [75]. These topics will, however, not be discussed in the present article. Further results and discussion of these matters are found in [76-79]

At about the same time V. Cantoni [80] presented a clarifying paper, writing:

"It is shown, *on purely kinematical grounds*, that one of the assumptions implicitly contained in the statement of Ehrenfest's paradox is not correct, the assumption being that the geometry of Minkowski space-time allows the passage of the disk from rest to rotation in such a fashion that both the length of the "radius" and the length of the periphery, measured with respect to the co-moving frame of reference, remain unchanged.

The following discussion is believed to show that a careful definition of all quantities involved eliminates the paradox and, with it, the alleged inconsistency of relativistic kinematics.

At any rate, should relativistic kinematics not be self-consistent, it would seem hard, on logical grounds, to accept the view that the addition of dynamical arguments might improve the situation."

Cantoni then presented the following analysis with reference to fig.5.

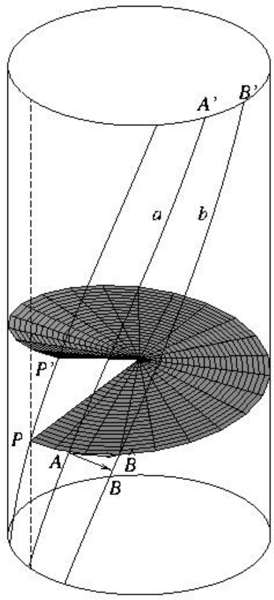


Fig.5. The helices such as *a* and *b* represent world lines of points at rest in the rotating frame. Points at rest in the inertial rest frame of the axis are represented by vertical world-lines. Cantoni's figure has here been modified by drawing not only the circumference defined by simultaneity in the local inertial rest frames of the disk material, but also the surface connecting the circumference with the axis.

"One can give a consistent definition of the length of the whole "circumference" relative to the rotating reference frame *K* as the length of the curve *PP'* everywhere orthogonal to the world-lines of the reference points of *K*, starting from an event *P* and ending on an event *P* on the world-line of the same particle of the edge of the disk, and winding once around the axis of rotation. *Notice that such a curve is not closed* in space-time, and distant events on it, such as

P and *P’*, can in no sense be regarded as simultaneous. For the segment between two infinitely close particles with world-lines, say, *a* and *b*, denoting by dl the length relative to the inertial rest frame *K* of the axis (i.e., the length of $\overset{\frown}{AB}$), and by $d\hat{l}$ the length relative to *K* (i.e., the length of $\overset{\frown}{A\hat{B}}$), the latter being the same as the length of the segment with respect to its local inertial rest-frame, one has, according to the well known equation for the Lorentz contraction,

$$d\hat{l} = dl\left(1 - v^2 / c^2\right)^{-1/2} \tag{78}$$

Since $\overset{\frown}{AB} = \overset{\frown}{A'B'}$ dl can also be interpreted as the length of the arc of circumference $A'B'$ at a fixed time, say $t = 0$ in *F* , and one has, with obvious notations,

$$d\hat{l} = \left(1 - v^2 / c^2\right)^{-1/2} r\, d\theta \tag{79}$$

so that, integrating around the circumference, one gets for the length $\hat{L}$ of the edge relative to *K*

$$\hat{L} = 2\pi r\left(1 - v^2 / c^2\right)^{-1/2} > 2\pi r \tag{80}$$

Clearly it was not justified to assume *a priori* that both the length of the radius of the disk and the length of its periphery, as measured from the co-moving frame of reference, could remain unchanged in the passage from rest to rotation.

In other words, In Minkowski space-time there exists no *rigid* family of world-lines describing the passage of a disk from rest to rotation: any family of world-lines describing a still disk during an initial stage and a rotating disk during a final stage is *necessarily non-rigid* in the intermediate stage. This fact only depends on the geometrical structure of Minkowski space and on the definition of rigidity, and is not a consequence of the dynamical properties of actual materials.”

G. Rizzi and A. Tartaglia [81] stressed the significance for the analysis of the Sagnac effect of the fact that the periphery of the disk as defined by simultaneity in the instantaneous inertial rest frames of the elements of the periphery is discontinuous in spacetime, as shown in Fig. 4.

W. A. Rodrigues [82] did not accept the possibility of a purely kinematical solution to Ehrenfest’s paradox. He returned to a pre-relativistic view of the ether,

arguing that the Lorentz contraction is a real phenomenon which results as a consequence of the interaction of material bodies with the ground-state vacuum of the universe.

This conception when applied to the Ehrenfest paradox implies that the circumference of the rotating disk gets a contraction due to the interaction of the periphery with the physical vacuum. Measuring rods along the periphery will also contract. Hence the measured length of the periphery is independent of the rotational velocity, and the geometry remains Euclidean.

According to Rodrigues there is no tangential strain, and hence no stress, in the rotating disk because its length has not been changed as the disk were put into rotation. A similar view was held by Phipps [83] which resulted in a discussion with Cantoni [84, 85] that did not, however, clarify the matter very much. Grøn [86] responded to Phipps' article by giving a covariant formulation of Hooke's law [87]. If this is accepted the existence of strain is invariant. The strain is defined with reference to the rest length of a mass element, so that a material becomes strained if the rest length changes. According to the covariant formulation of Hooke's law this leads to stress in the material. Thus, if a disk is put into rotation by accelerating all points identically as measured in K', then the length of the periphery remains unchanged in K', but its rest length increases and the material will get a tangential stress [88]. D. Dieks [89] pointed out that the existence of such stresses demonstrates the physical nature of the Lorentz contraction, even when interpreted within the special theory of relativity.

Winterberg [58] presented an "ether-interpretation" of the Lorentz contraction and "solved" the Ehrenfest paradox by calculating the deformation of the disk when it is put into rotation. He also suggested an experiment to test special relativity against the "dynamic Lorentz-Poincaré interpretation of the Lorentz contraction. E. M. Kelly [90] introduced a new sort of contraction (or expansion) for moving bodies to solve the Ehrenfest paradox. R. D. Klauber [91] gave an analysis of space and time on a rotating disk based upon the assumption that the special theory of relativity is not applicable to rotating frames. He replaced Einstein's postulate on the isotropy and invariance of the velocity of light by a postulate saying that "The speed of light is not invariant between the ground and the rotating frame, and in the rotating frame is found to first order by the velocity addition law $\left|v_{\text{tangential, light}}\right| \cong c \pm r\omega$." Then he deduced among other things an Euclidean spatial geometry on the rotating disk. A discussion about these topics between Klauber and Weber appeared in American Journal of Physics [92, 93]. Further discussion of the length of the circumference of a rotating disk with radius R, maintaining that it is $2\pi R$, was presented by A. Tartaglia [94].

The metric in a rotating frame as solution of Einstein's field equations

In 1951 there appeared a somewhat surprising article [95] by B. Kurşunoğlu titled *Space-time on the Rotating Disk*. There are 3 unusual properties of its contents.

I. He wrote the metric in the rotating frame in diagonal form

$$ds^2 = -e^{\nu} dt^2 + dr^2 + e^{\mu} d\theta^2 + dz^2 \tag{81}$$

where μ and ν are functions of r alone.

II. The field equations were solved with the energy-momentum of an elastic medium as source. But the density and pressure of the medium disappeared from the equations and are not present in the solution, which is given as follows

$$ds^2 = -\left(c^2 - r^2\omega^2\right)dt^2 + dr^2 + \frac{r^2 d\theta^2}{1 - r^2\omega^2 / c^2} + dz^2 \tag{82}$$

Here ω is the angular velocity of the rotating frame.

III. Kurşunoğlu then calculated the curvature of space-time and found a non-vanishing value. Hence, he concluded that "for an observer on a rotating disk there is no way of escape from a curved space-time".

Concerning point I it was shown by Weysenhoff [30] that it is not possible to cover a rotating frame with an orthogonal coordinate system. Hence the metric cannot be diagonal in a rotating coordinate system.

The points II and III were investigated by Grøn [96].

Let us first consider the point II. It was shown that the energy-momentum tensor specified by Kurşunoğlu could be represented by a perfect fluid. For a static metric of the form (81) the field equations then imply the equation of state $p = -(1/3)\rho$ (in this section we use units so that $c = G = 1$). Solving the field equations with mass density ρ_0 at the axis, one finds the solution

$$ds^2 = -\left(1 + \frac{8\pi\rho_0}{3} r^2\right)dt^2 + dr^2 + \frac{r^2 d\theta^2}{1 + \frac{8\pi\rho_0}{3} r^2} + dz^2 \tag{83}$$

Introducing curvature coordinates by the transformation $\hat{r} = r\left(1 + \frac{8\pi\rho_0}{3} r^2\right)^{-1/2}$ the line element takes the form

$$ds^2 = -\frac{dt^2}{1-\frac{8\pi\rho_0}{3}\hat{r}^2} + \frac{d\hat{r}^2}{\left(1-\frac{8\pi\rho_0}{3}\hat{r}^2\right)^3} + \hat{r}^2 d\theta^2 + dz^2 \tag{84}$$

The calculation shows that the interpretation of eq.(82) that it represents the metric of empty space in a rotating coordinate system is not viable.

On the other hand, if one starts with a stationary, cylindrically symmetric line-element of the form

$$ds^2 = f(r)dt^2 + dr^2 + l(r)d\theta^2 + dz^2 + 2m(r)dt\,d\theta \tag{85}$$

and solves the vacuum field equations, one finds the line element (7).

We now go on to point III. As pointed out by Wilson [97] the spacetime described by the metric (7) is flat. However, if one calculates the spacetime curvature from the line element (82) one finds $R_{\theta\theta} = -3r^2\omega^2\left(1-r^2\omega^2\right)^{-3}$, which was the reason for the conclusion of Kurşonoğlu cited above. However, the correct interpretation of this expression is obtained by writing it as

$$R_{\theta\theta} = \frac{32\pi\rho_0}{3}r^2\left(1+\frac{8\pi\rho_0}{3}r^2\right)^{-3} \tag{86}$$

This represents a component of the Ricci curvature tensor in a static spacetime filled with an elastic medium.

Finally it may be noted that one may obtain a line-element that is formally identical to eq.(82) by introducing a non coordinate basis of one-forms $\left(\boldsymbol{\omega}^{\mathbf{t}}, \boldsymbol{\omega}^{\mathbf{r}}, \boldsymbol{\omega}^{\boldsymbol{\theta}}, \boldsymbol{\omega}^{\mathbf{z}}\right)$ given by

$$\boldsymbol{\omega}^{\mathbf{t}} = \mathbf{d}t' - r'^2\,\omega\left(1-r'^2\,\omega^2\right)^{-1}\mathbf{d}\theta', \quad \boldsymbol{\omega}^{\mathbf{r}} = \mathbf{d}r', \quad \boldsymbol{\omega}^{\boldsymbol{\theta}} = \mathbf{d}\theta - \omega\mathbf{d}t', \quad \boldsymbol{\omega} = \mathbf{d}z' \tag{87}$$

where $\left(\mathbf{d}t', \mathbf{d}r', \mathbf{d}\theta', \mathbf{d}z'\right)$ are the coordinate basis forms of the non-rotating cylindrical coordinate system. In this basis the line element takes the form

$$ds^2 = -\left(1-r^2\omega^2\right)\left(\omega^t\right)^2 + \left(\omega^r\right)^2 + \frac{r^2\left(\omega^\theta\right)^2}{1-r^2\omega^2} + \left(\omega^z\right)^2 \tag{88}$$

Kurşunoğlu writes the metric in this form with dt instead of $\boldsymbol{\omega}^{\mathbf{t}}$ and so forth. This metric has also been considered by Adler et al. [98] and essentially the same metric by Arzeliès [99] who also calculated a non-vanishing spacetime curvature. However, the vanishing of the curvature of the Minkowski spacetime is invariant, and cannot be changed by expressing the metric in a new basis. The reason for their result is that both Arzeliès and Kurşunoğlu treated the metric as if it was expressed in a coordinate basis. However, this is not the case since $\boldsymbol{\omega}^{\mathbf{t}}$

is not an exact differential form. This means that the curvature tensor cannot be calculated by means of the usual expressions valid for a coordinate basis. One has to include the structure coefficients in the calculation. Making this one obtains zero curvature.

Kinematical solution of Ehrenfest's paradox

Sama [100] has argued that this paradox only arises from an ambiguous use of notation. However, there are deep and interesting points connected by the problem raised by Ehrenfest [1] that cannot be analysed by restricting oneself to notational matters. One is the role of the relativity of simultaneity which is essential to obtain a kinematical solution of the paradox. This solution which will be reviewed in the present section, was given by Grøn several years ago [101].

Assume that there are n marks on the circumference of a disk which is rotating with an angular velocity ω. One wants to increase the rate of rotation by giving the marks small blows. In order to increase the angular velocity in a Born rigid way the blows must be given to the marks simultaneously in their instantaneous inertial rest frames.

We now consider this acceleration program from the point of view of the inertial rest frame of the axis. The marks are enumerated from 1 to n in the direction of the rotation. Performing Lorentz transformations from the instantaneous inertial rest frames of the marks to the laboratory frame, one finds that 2 gets a blow later than 1, 3 later than 2, and so forth. Going around the disk one finds that n happens later than n-1. Hence, n happens later than 1. But passing on from n to 1, the blow 1 should happen later than n since the events should be simultaneous as measured in the instantaneous rest frame of the element between the marks n and 1. P. Noerdlinger [102] has provided a graphical illustration of this fact, commenting:

> Observers following a rotating ring will not synchronize their clocks in the same way as inertial ones. Cumulating this result around the ring, leaves a time mismatch depending on the rotational speed and area enclosed.

This shows that due to the relativity of simultaneity the acceleration program that would represent a Born rigid increase of the angular velocity of the disk, define a set of kinematically self contradictory boundary conditions. The conclusion is that a Born rigid transition of the disk from rest to rotation is a *kinematic* impossibility. This was mentioned already in 1921 by Pauli [103] who wrote:

> A simple argument by Ehrenfest [1] shows however that such a body [Born rigid] cannot be set in rotation.

Energy associated with tangential stress in a rotating disk

M. H. Mac Gregor [104] has questioned the existence of relativistic stresses in a rotating ring. He argued as follows:

Let us fasten a string between two points *A* and *B* on the circumference of a disk, and then give the disk an angular acceleration, so that it attains an angular velocity ω. An observer in the disk co-ordinate frame *K* who studies this event sees the distance *AB* increase with increasing angular velocity ω, as specified by eq.(10), and he concludes that the string has become stretched. An observer in the fixed inertial system K' who studies this same event sees the distance *AB* as remaining unchanged. (since the points *A* and *B* has similar acceleration histories), but he knows that the string extending from *A* to *B* has undergone a relativistic contraction in length; hence he also concludes that the string has become stretched. This stretching is a purely kinematic result, and it gives rise to relativistic stress. The observers in *K* and K' each deduce a kinematic stretching of the string by a factor of $\left(1-a^2\omega^2/c^2\right)^{-1/2}$, where a is the radius of the disk. Not only is the string stretched by a factor of $\left(1-a^2\omega^2/c^2\right)^{-1/2}$ when the disk is accelerated to an angular velocity ω, but the mass of the string is also increased by this same factor. Hence the mass per unit length of the string remains unchanged. If we measure relativistic stresses in terms of changes in linear *density* (mass per unit length) rather than in terms of simple changes in *length*, then these stresses do not occur.

In this way Mac Gregor defines away the relativistic stress associated with the Lorentz contraction of the material. However the stress cannot be removed by a definition. It has physical effects. For example, by velocities of the periphery sufficiently close to the velocity of light, the disk material would crack. There is an energy associated with the relativistic stresses, and this implies that one must perform an extra amount of work in order to change the angular velocity of a disk due to these stresses. It was shown by Grøn [105] how this comes about due to the relativity of simultaneity.

We shall consider retardation of a rotating ring consisting of small springs with elastic constant k and rest length L_0. Initially the ring rotates so that the springs are close to each other, end to end, but without stress. We assume that the braking is performed by simultaneous blows around the ring, as measured in K'. Then the time difference of two blows at the ends of a spring, as measured in the instantaneous rest frame K_0 of the spring is

$$\Delta t = \gamma\left(a\omega/c^2\right)L \tag{89}$$

where $L = \gamma^{-1}L_0$ is the initial Lorentz contracted length of the spring. If the velocity change of the rear end of the spring is $dv' = a\,d\omega$ as observed in K', it follows from the Lorentz transformation that the velocity change in K_0 is

$$dv = -\gamma^2 a\,d\omega \tag{90}$$

During the time interval Δt the rear point moves towards the front point with this velocity as observed in K_0. So the spring gets a compression

$$ds = dv\,\Delta t = -\gamma^3 \left(a^2\omega / c^2\right) L\, d\omega \tag{91}$$

Integration gives

$$s = \left(\gamma_0 - \gamma\right) L \tag{92}$$

where $\gamma_0 = \left(1 - a^2\omega_0^2 / c^2\right)^{-1/2}$.

The force acting on the spring in order to compress it is ks. Hence a work is performed on the spring,

$$W = \int_{\omega_0}^{0} ks\, ds = \frac{1}{2} k \left[s^2 \right]_{\omega_0}^{0} = \frac{1}{2} kL^2 \left(\gamma_0 - 1\right)^2 \tag{93}$$

where we have used eq.(92). As expressed by the initial uncompressed rest length of the spring the work is

$$W = \frac{1}{2} kL_0^2 \left(1 - \gamma_0^{-1}\right)^2 \tag{94}$$

This work, which has been calculated in the instantaneous inertial rest frame of a spring, gives the contribution to the spring's rest mass, due to stresses developed when the angular velocity of the ring is changed in a Born rigid way. Expanding in powers of $a\omega / c$ and retaining only the first term gives

$$W \cong \left(a\omega_0 / 2c\right)^4 2kL_0^2 \tag{95}$$

This shows that the accumulated potential energy in a rotating ring due to relativistic stresses is a fourth order effect in v / c.

A rotating disk with angular acceleration

M. Strauss [106] claimed that:

> If the measuring rods laid along the circumference of the rotating disk are Lorentz contracted with respect to the inertial frame, so are the distances on the circumference they are supposed to measure; hence the two effects would cancel each other, and the ratio C/D (circumference/diameter) would turn out to equal π as in the Euclidean plane.

Grøn [107] has argued that this claim cannot be true. Firstly, as shown in the previous section, the disk cannot be put into rotation in a Born rigid way. Hence, the circumference will not be Lorentz contracted.

We now assume that the circumference of the disk is covered by n standard measuring rods. A *standard* measuring rod in a reference frame with arbitrary motion is has by definition constant proper length. As observed from a reference frame where they move standard mesuring rods are subject to Lorentz contractions, only, which means that they perform Born rigid motions.

In order to obtain this n rods are assumed to rest on the disk without friction, being kept in place by a frictionless rim on the circumference of the disk, each rod being fastened to the disk at one end only, at points k so that they just cover the circumference when the disk is not rotating, as shown in figure 6.

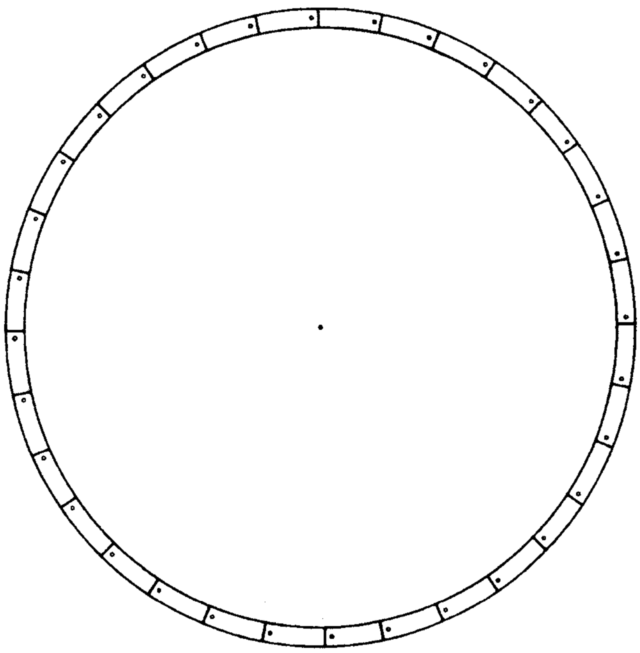


Fig.6. The disk and the measuring rods at rest.

The only isotropic way of giving the disk an angular velocity is to accelerate all points of the disk simultaneously as measured in the laboratory frame K'. In the rest frame K_k of a point k on the periphery of the disk one then measures that the point k is accelerated at a point of time

$$\Delta t_k = \left(1 - a^2\omega^2 / c^2\right)^{-1/2} \left(a\omega / c^2\right) L_0 \tag{96}$$

earlier than the point k+1, where $L_0 = 2\pi a / n$ is the proper length of the standard measuring rod at k. Thus the distance between these points, that is the point at the front of one mesuring rod and the front of the next, increases as observed in K_k. However, according to their definition the standard measuring rods move rigidly. Their proper length remain unchanged.

Accordingly the rods separate from each other as the angular velocity of the disk increases. If the velocity of a rod is increased from $a\omega$ to $a(\omega + d\omega)$ as observed in K', its velocity change, as observed in K_k, is $\left(1 - a^2\omega^2 / c^2\right)^{-1} a\, d\omega$. During this change the distance between two neighbouring rods increases with

$$ds_k = \left(1 - a^2\omega^2 / c^2\right)^{-3/2} \left(a^2\omega / c^2\right) L_0\, d\omega \tag{97}$$

Integrating, one finds the distance between the rods, as measured in K_k, when the disk rotates with an angular velocity ω

$$s_k = \left[\left(1 - a^2\omega^2 / c^2\right)^{-1/2} - 1\right] L_0 \tag{98}$$

Thus the distance as measured in K is

$$s = L_0 - L_0 \left(1 - a^2\omega^2 / c^2\right)^{1/2} \tag{99}$$

in accordance with the fact that the measuring rods are Lorentz contracted, while the circumference of the disk is not. The observation in K' of the rotating disk and the measuring rods is shown in Fig.7.

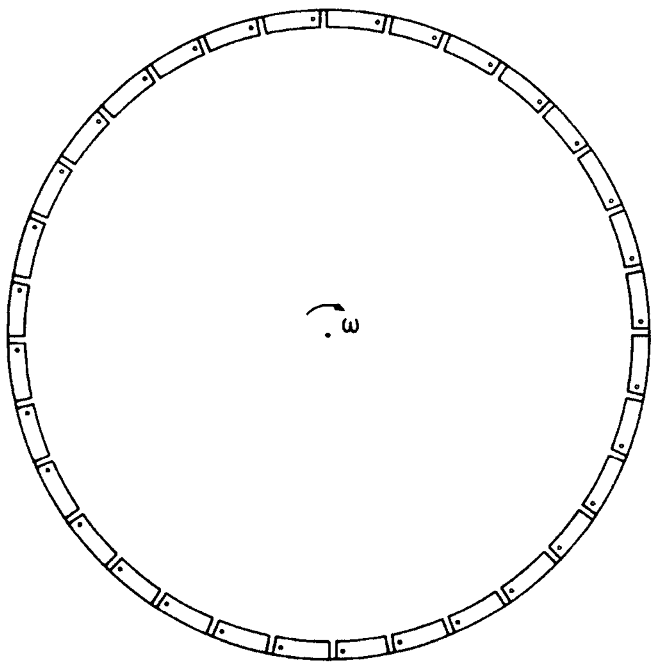


Fig.7. The rotating disk and the measuring rods.

The results of this analysis imply that the ratio of the length of the circumference of a rotating disk and the length of its diameter, measured by means of standard measuring rods at rest on the rotating disk is $\left(1 - a^2\omega^2 / c^2\right)^{-1/2} \pi$. Since this statement is invariant under a transformation connecting two different coordinate systems inside the same system of reference, as it is based on the use of standard instruments, it is a statement characterising the intrinsic spatial geometry of the rotating disk. It follows that this geometry is non-Euclidean.

A similar result has been found by considering a rotating disk with angular acceleration [108].

A rolling disk

We shall here follow the presentation of Kevin Brown [109]. A slightly different analysis of a rolling disk was given by Grøn [101].

A disk with radius a is rolling so that the axis moves with constant velocity $a\omega$ along the negative X-axis in the instantaneous inertial frame K_0 of that element of the disk which has contact with the ground. The co-moving coordinates of this frame are (T, X, Y). A fixed point on the disk at the location (r,θ) has at the time t' the coordinates (t', x', y') in the rest frame K' of the axis,

$$t'(r,\theta,t') \ , \quad x'(r,\theta,t') = r\cos(\omega t' + \theta) \ , \quad y'(r,\theta,t') = r\sin(\omega t' + \theta) \qquad (100)$$

Making a Lorentz transformation to the laboratory system leads to

$$T(r,\theta,t') = \gamma\left(t' - r\omega x'/c^2\right) = \gamma\left[t' - \left(r^2\omega/c^2\right)\cos(\omega t' + \theta)\right] \ , \ \gamma = \left(1 - r^2\omega^2/c^2\right)^{-1/2} \quad (101A)$$

$$X(r,\theta,t') = \gamma\left(x' - r\omega t'\right) = \gamma\left[r\cos(\omega t' + \theta) - r\omega t'\right] \ , \quad Y(r,\theta,t') = r\sin(\omega t' + \theta) \quad (101B)$$

We shall determine the position of the point on the disk at a given point of time in the laboratory system, say $T = 0$. From the equation for $T(r,\theta,t')$ we see that the instant $T = 0$ in K_0 correspond to different points of time in K' given by

$$t' = \left(r^2\omega/c^2\right)\cos(\omega t' + \theta) \qquad (102)$$

Substituting this into eq.(101) gives

$$\gamma X(r,\theta)_{T=0} = r\cos\left[\omega t'(r,\theta) + \theta\right] \ , \quad Y(r,\theta)_{T=0} = r\sin\left[\omega t'(r,\theta) + \theta\right] \qquad (103)$$

Hence the circumference $r = a$ is given by

$$(\gamma X)^2 + Y^2 = a^2 \qquad (104)$$

which describes an ellipse with half-axis a in the Y-direction and half-axis $\gamma^{-1}a$ in the X-direction.

Radial lines of the disk, given by $\theta =$constant, appear as curved by simultaneity in the laboratory frame. This is shown in Fig.8.

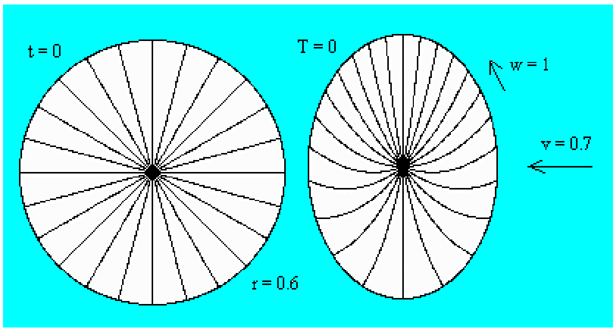


Fig.8. The non-rotating disk is drawn to the left and the rolling disk to the right. Radial lines of the rolling disk are curved as observed by simultaneity in the frame K_0.

Some complementary figures showing the appearance of the rotating disk in accordance with two operational procedures performed in the rotating rest frame of the disk, have been presented by K. MacFarlane [110].

The positions of points on a rolling ring at retarded points of time were calculated with reference to K_0 by Ø. Grøn [111]. The result is shown in Fig. 9. Part C of the figure shows the "optical appearance" of a rolling ring, i.e. the positions of emission events where the emitted light from all the points arrives at a fixed point of time at the point of contact of the ring with the ground. In other words it is the position of the points when they emitted light that arrives at a camera on the ground just as the ring passes the camera.

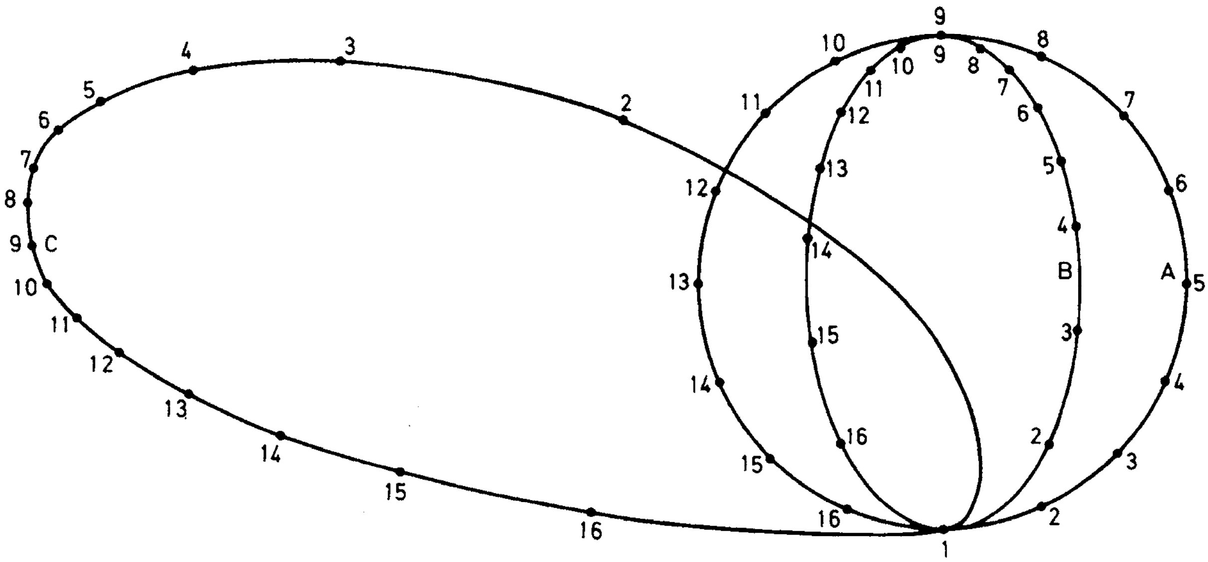


Fig. 9. Points on a rolling ring. A: Observed by simultaneity in its rotating rest frame K; B: observed by simultaneity in the frame K_0: C: observed in K_0 at retarded points of time.

An interesting observation concerning a rolling disk was made by K. Vøyenli [112]. He calculated the length of the circumference by considering the distance on the ground between marks of a fixed point 0 on the circumference, as the disk rolls with constant velocity.

The period *T* of revolution relative to K_0 of a point *P* on the periphery is according to the time dilation formula given by $T = \gamma T_0$. We find accordingly that the point 0 in one revolution of the periphery moves a distance *L* relative to K_0 given by $L = vT$ or

$$L = \gamma 2\pi a \tag{105}$$

The same result has recently been obtained by Grøn [101] in a different connection and in a less simple way.

The distance *L* may be called the "rolled out" circumference of the disk and may be identified with the proper circumference L' by the following argument. We consider a division of the periphery of the disk into infinitesimal line elements by points fixed om the periphery. These elements are matched one to one with line elements on the x-axis, such that corresponding elements coincide at the moment the element on the periphery touches the x-axis and is momentarily at rest in K_0. We make the usual assumption that an accelerated standard measuring rod, which momentarily is at rest in a given inertial frame, may coincide at this moment with an identical rod permanently at rest in the same inertial frame. It follows that corresponding elements on the periphery and the x-axis must have the same length relative to the rotating frame *K* and the inertial frame K_0, respectively, and that the proper circumference of the disk must be equal to the "rolled out" circumference.

The rotating disk and the Thomas precession

D. H. Weinstein [113] claimed that the extension of a disk will be reduced, and radial lines deformed, as shown in Fig.10, due to the Thomas precession when a disk is put into rotation.

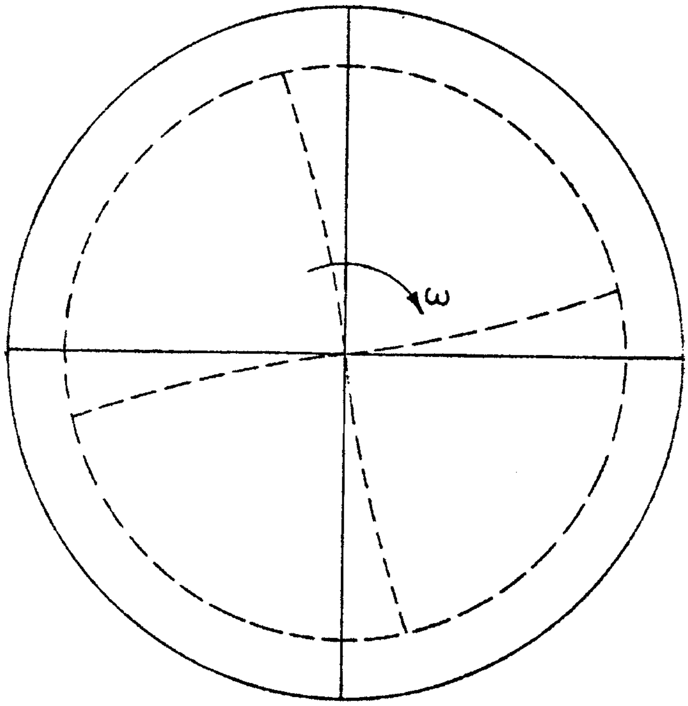


Fig.10. The rotating disk at rest is shown solid drawn and the rotating disk is shown with dashed line.

The Thomas precession is a kinematical special relativistic effect appearing if a rod which is free to rotate moves along a circular path normally to its axis of rotation. However, when a disk is put into rotation, say by giving all points on the circumference equal and simultaneous tangential blows, the motion of the disk material is determined mainly by the force acting on the disk and the elastic properties of the material, as was emphasised by G. Cavalleri [114]. Also, it was pointed out by Newburgh [115] that although the Thomas precession is clearly significant in the case of spinning elementary particles moving along curved paths, is by no means clear that the material of an extended body will undergo a Thomas precession.

The effect of the Thomas precession would be an accumulating strain which would cause a tension in the material. This tension would rapidly counteract further strain. Hence the effect of the Thomas precession would vanish practically immediately. It would therefore not be possible to measure any accumulated "Thomas strain" for a disk having rotated 1000 times per second for 30 days, as suggested by Weinstein.

This was shown very clearly by Whitmire [116] who wrote:

> Consider two elements S_1 and S_2 in a thin spinning disk. For definiteness we consider circular regions of the same size and the same radial position *r*. Let us assume that S_1 and S_2 undergo a Thomas rotation given by
>
> $$\omega_T = (\gamma - 1)\frac{\mathbf{v}\times\mathbf{a}}{\mathbf{v}^2} \quad , \quad \gamma = \left(1 - v^2/c^2\right)^{-1/2} \quad , \quad \mathbf{v} = \omega\times\mathbf{r} \tag{106}$$
>
> where ω is the angular velocity of the disk, and $\mathbf{a}$ is the centrifugal acceleration $|\mathbf{a}| = v^2/r$. The sense of ω_T is opposite to ω. Now consider the case where S_1 and S_2 as in Fig.6. Particles in the overlap region (shaded area in Fig. 11) must shear the motion of S_1 and S_2 simultaneously. In other words, the matter in the overlap region must move in two (opposite) directions at once if it is assumed that eq.(106) applies to an arbitrary section of mass in the disk.

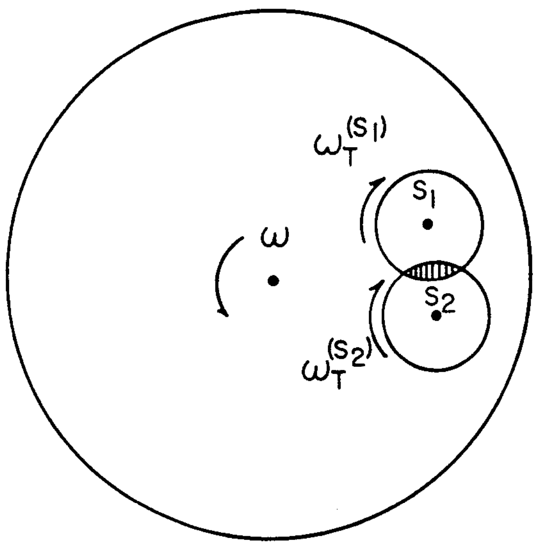


Fig.11. Thomas rotation of two overlapping circular regions.

> The problem of the Thomas rotation in a spinning macroscopic disk can be resolved by consideration of the elastic properties of the disk. Consider a small element of mass in the rotating disk. If it is uncoupled (or weakly coupled) to the rest of the disk then it will undergo the usual Thomas rotation. If it is strongly coupled to the rest of the disk, however, the mass elements will be constrained from rotating, thus introducing "Thomas shear stresses". In a laboratory experiment with a rotating disk there will be no Thomas rotation of segments in the disk.

Contracted rotating disk

Grünbaum and Janis [117] have considered a disk put into rotation in such a way that the radius contracts and no tangential stresses appear. This means that the rest length of tangential mass elements remains unchanged during the period of angular acceleration. At first moment one might think that this is not possible. Due to the relativity of simultaneity the special theory of relativity forbids, in the case of rotating motion with constant radius, to increase the angular velocity of a rotating disk in a Born rigid way. Hence, tangential stresses will appear, and the rest length of the periphery changes. There was a discussion of this in Foundations of Physics [118, 119] and it became clear that the type of motion considered by Grünbaum and Janis is indeed permitted by the theory of relativity.

The main result of Grünbaum and Janis' can be deduced as follows. Allowing for a changing radius of the disk the coordinate transformation between K and K' may be written

$$t' = t \ , \quad r' = f(r) \ , \quad \theta' = \theta + \omega t \tag{107}$$

Inserting this into the general expression (17) for the spatial line element leads to

$$d\sigma^2 = (df/dr)^2 \, dr^2 + \left\{ f^2(r) / \left[1 - f^2(r)\omega^2/c^2\right]^{1/2} \right\} d\theta^2 \tag{108}$$

Consider a point on the disk. Before the rotation was started it had a radial coordinate $r_1' = r$ in K'. In the final state of uniform rotation it has a radial coordinate $r_2' = r'$ in K' and $r_2 = r$ in K.

The function f is determined from the condition that there are no tangential strains in the disk in the final state as compared to the non rotating state. This means that the rest length of a given circle on the disk is the same in the final state as in the non rotating state,

$$\sigma_\theta = \left(\gamma_{\theta\theta}\right)_r^{1/2} 2\pi = 2\pi r \tag{109}$$

Hence $d\sigma_\theta = r\, d\theta$. This, together with eq.(108), gives

$$f(r) / \left[1 - f^2(r)\omega^2/c^2\right]^{1/2} = r \tag{110}$$

which leads to

$$f(r) = \frac{r}{\sqrt{1 + r^2\omega^2/c^2}} \tag{111}$$

Hence, from eq.(107) we have

$$r' = \frac{r}{\sqrt{1 + r^2\omega^2/c^2}} \tag{112}$$

This equation the radius r' of a circle on the rotating disk as compared to the radius of the same circle when the disk does not rotate. It shows that the disk contracts when the angular velocity increases.

Substituting eq.(111) for $f(r)$ into eq.(108) gives the spatial line element of the rotating disk

$$d\sigma^2 = \left(1 - r^2\omega^2 / c^2\right)^{-3} dr^2 + r^2 d\theta^2 \tag{113}$$

This line-element shows that the circumference of the disk is $2\pi r$, independently of the angular velocity, and the spatial geometry is negatively curved since $circumference / diameter > \pi$. The proper radial distance is given by

$$d\sigma_r = \left(\gamma_{rr}\right)^{1/2} dr = \left(1 + r^2\omega^2 / c^2\right)^{-3/2} dr \tag{114}$$

Integration gives

$$\sigma_r = \frac{r}{\sqrt{1 + r^2\omega^2 / c^2}} \tag{115}$$

which also exhibits the contraction of the rotating disk. Expressing the spatial line element of the rotating disk in terms of the proper radial distance one obtains

$$d\sigma^2 = d\sigma_r^2 + \frac{\sigma_r^2 d\theta^2}{1 - \sigma_r^2\omega^2 / c^2} \tag{116}$$

which has the same form as the line element (10).

Since the proper length of the circumference does not change when the disk is put into rotation, one might think that the acceleration program that realises this motion might represent self-contradictory boundary conditions due to the relativity of simultaneity, as is the case for purely tangential motion. However, the motion can be produced in the following way. The disk is initially compressed in accordance with eqs.(112) and (115). Then all points of the disk are accelerated in the tangential direction by a succession of blows, each blow being given to all points simultaneously in the inertial rest frame K' of the axis.

The disk motion introduced by Grünbaum and Janis has later been considered by Ziino [120] with a slightly different interpretation. He deduced the expression (115) and the line elements (113) and (116) and gave the following comment (using the notation above):

> What can essentially be gained is a more orthodox (and still relativistically consistent) geometrical definition of a rotating frame, in terms of a suitable "world" radial co-ordinate that may *naturally run to infinity,* with no need for values greater than c / ω to be ruled out. The new radial coordinate, *r* , differs from the standard one, σ_r , by the following: it is identically equal to the *Euclidean* radius, $\sigma_r \gamma\left(\omega\sigma_r\right)$,

of a circumference of proper length $2\pi\sigma_r\gamma\left(\omega\sigma_r\right)$ which is described in the rotating frame at an actual radial distance σ_r from the rotation axis. A "new" metric could accordingly be assigned to a rotating frame, which can be obtained by just recasting the usual metric in terms of *r*. The most immediate physical application concerns the kinematics of a uniformly spinning disk (with presumably far-reaching effects on the physics of rotating black holes). The result is that a disk of *whatever* (original) radius *r* might be brought to spin with an *arbitrarily great* uniform angular velocity ω: its shape should not undergo any distortion with spinning, but should appear to be *globally contracted* by a scale factor $\gamma^{-1}\left(\omega\sigma_r\right)$, where $\omega\sigma_r = \omega r\left(1+\sigma_r^2\omega^2/c^2\right)^{-1/2}$ and σ_r is the *new* radius that the disk would exhibit when it is seen rotating with an angular velocity ω.

Conclusion

There are several results of the long period with discussions on the geometry of a rotating disk.

1. Ehrenfest's paradox demonstrated that it is not possible to put a disk into rotating motion in a Born rigid way while remaining horizontal.
2. Einsteins's argument based on using standard measuring rods on a rotating disk to measure it geometrical properties shows that the periphery of a disk with radius r rotating with angular velocity ω has a length $2\pi r/\sqrt{1-r^2\omega^2/c^2}$. Hence, it is longer than $2\pi r$, not shorter as in the formulation of Ehrenfest's paradox. With reference to the inertial rest frame of the axis this is explained as due to the Lorentz contraction of the measuring rods in the tangential direction and not in the radial direction. With reference to the rotating rest frame of the disk it is interpreted as a gravitational effect, i.e. the geometry of space is non-Euclidean in a gravitational field.
3. Due to the relativity of simultaneity Born rigid rotating motion of a ring with angular acceleration represents contradictory boundary conditions.
4. If the disk is regarded as a 2-dimensional surface it can be put into rotation in a Born rigid way, that is without any displacements in the tangential plane of the disk, by bending for example upwards so that it obtains the shape of a cup.
5. The surface orthogonal to the world lines of the disk particles is called the 3-space in the rotating rest frame of the disk. This space is defined by events that are simultaneous as measured by Einstein synchronised clocks on the disk. It has a discontinuity along a radial line as shown in fig.5, and is negatively curved.
6. Spatial geodesics curve inwards the 3-space of a rotating reference frame. This demonstrates the negative curvature of this space.
7. One may introduce local coordinates in the neighbourhood of arbitrary points on a rotating disk by means of differential transformations from coordinates in the inertial

rest frame of the axis. These transformations may be chosen so that the spatial line element at constant time in the rotating system has Euclidean form. Also one may calculate a non-vanishing Riemann curvature tensor for spacetime in the rotating frame by employing the usual formulae valid in a coordinate basis. This does not mean, however, that the 3-space is flat and spacetime is curved in the rotating frame. Taking account of the non-vanishing structure coefficients in a non-coordinate basis one finds that the 3-space is curved and spacetime is flat in the rotating frame.

8. What actually happens when a disk is put into rotation depends upon its elastic properties. A maximally rigid disk, with sound velocity equal to the velocity of light, will in fact contract when its angular velocity in increased.

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