Graphon games: A statistical framework for network games and interventions

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Motivation

Social interactions
- Adoption of innovations, behaviors
- Opinion formation
- Social learning

Economic interactions
- Public good provision
- Competition among firms
- Financial trades

In many social and economic settings, decisions of individuals are affected more by the actions of their friends, colleagues, peers and competitors.
Network game model

Consider a network game defined by:
- $N$ agents
- interacting over a network $G \in \mathbb{R}^{N \times N}$
  \[
  \begin{cases}
  G_{ij} \geq 0 & \text{influence of } j \text{ on } i \\
  G_{ii} = 0 & \text{no self loops}
  \end{cases}
  \]

Each agent $i$ aims at minimizing its cost function

- strategy: $x^i \in \mathbb{R}^n$
- feasible set: $\mathcal{X}^i \subset \mathbb{R}^n$
- cost: $J^i(x^i, z^i(x)) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$
- aggregate: $z^i(x) := \frac{1}{N} \sum_{j=1}^{N} G_{ij} x^j$

Standing assumption

- $\mathcal{X}^i \subset \mathbb{R}^n$ compact and convex;
- $J^i(x^i, z^i(x))$ strongly convex in $x^i$, for all $x^{-i} \in \mathcal{X}^{-i}$;
- $J^i(x^i, z^i) \in C^2$ in $[x^i; z^i]$. 
• Each agent chooses **an action** $x^i \geq 0$ ... → how much effort exerted on an activity (e.g. education, smoking, public goods)

• **Agent $i$ cost function:**

$$J^i(x^i, z^i(x)) = \frac{1}{2} (x^i)^2 - a^i x^i - K \cdot z^i(x) \cdot x^i$$

- aggregate: $z^i(x) = \frac{1}{N} \sum_{j \neq i} G_{ij} x^j$
- $K$ determines how much neighbor actions affect agent's payoff. ($K > 0$ strategic complements; $K < 0$ strategic substitutes)

A set of strategies $\{\bar{x}^i\}_{i=1}^N$ is a **Nash equilibrium** if for each player $i$,

$$J^i(\bar{x}^i, z^i(\bar{x})) \leq J^i(x^i, z^i(\bar{x})), \text{ for all } x^i \in \mathcal{X}^i.$$
Literature and main question

What is the impact of network structure on equilibrium outcome?

- How does individual **network position** determine individual play?
  
  Ballester et al. (2006); Bramoullé and Kranton (2007); Bramoullé et al. (2014); Belhaj et al. (2014); Jackson and Zenou (2014); Acemoglu et al. (2015); Allouch (2015); Melo (2017); Parise and Ozdaglar (2018)

- How does a central planner **target interventions**?

  • Ballester et al. (2006): key-player removal in crime applications
  • Candogan et al. (2012): optimal pricing for monopolist
  • Galeotti et al. (2017): budget allocation in network games

  \[ \text{require exact network information} \]

Applications where network is large, changing over time or multiple networks

Can we regulate strategic behavior by using only **statistical information** about network interactions?
A statistical framework for network games

- Sampled network game = game over a sampled network

- Related work on distribution of centrality measures:
Graphons as stochastic network formation processes

Graphons - [Lovász, 2012]

A symmetric measurable function $W : [0, 1]^2 \rightarrow [0, 1]$

1) Limit of graph: $W(s, t) = \text{interaction } s, t \in [0, 1]$

$N = 5$  $N = 10$  $N = 20$  $N = \infty$
Graphons as stochastic network formation processes

**Graphons** -[Lovász, 2012]

A symmetric measurable function $W : [0, 1]^2 \rightarrow [0, 1]$

2) Random graph model:

Generalize **parametric models** such as Erdos-Renyi, Stochastic Block model
Graphons as stochastic network formation processes

**Graphons - [Lovász, 2012]**

A symmetric measurable function \( W : [0, 1]^2 \to [0, 1] \)

- **Theory:**
  [Lovász, Szegedy, 2006], [Lovász, 2012], [Borgs et al., 2008]

- **Applications:**
  - community detection [Eldridge et al., 2016],
  - crowd-sourcing [Lee and Shah, 2017],
  - signal processing [Morency and Leus, 2017],
  - optimal control of dynamical systems [Gao and Caines, 2017]
  - graphon mean field games: [Caines and Huang, 2018]
  - ...

- **Key idea of this work:**
  combine network game theory with graphon theory
Illustration for a SBM

Questions:
1. How close are Nash equilibria in different sampled network games?
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2. Will the Nash equilibria converge to a deterministic profile for $N$ large?
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1. How close are Nash equilibria in different sampled network games?
2. Will the Nash equilibria converge to a deterministic profile for $N$ large?
3. Can we exploit this property to design robust interventions for sampled network games?
Talk outline

Step 1 Define graphon games for infinite populations
- define equilibrium
- existence, uniqueness and sensitivity

Step 2 Relate infinite graphon games to sampled network games
- reformulate a network games as a step-function graphon game
- relate equilibria of graphon games & sampled network games
  (bound the distance in terms on the population size)

Step 3 Design interventions for sampled network games based on graphon model

Step 4 Incomplete information in sampled network games
Step 1:
Infinite population
Network versus graphon games

<table>
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<th>Network games</th>
<th>Graphon games</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agents</td>
<td>$i \in {1, \ldots, N}$</td>
<td>$s \in [0, 1]$</td>
</tr>
<tr>
<td>Interactions</td>
<td>$G \in \mathbb{R}^{N \times N}$</td>
<td>$W : [0, 1]^2 \rightarrow [0, 1]$</td>
</tr>
<tr>
<td>Strategy</td>
<td>$x^i \in \mathcal{X}^i$</td>
<td>$x(s) \in \mathcal{X}(s)$</td>
</tr>
<tr>
<td>Cost function</td>
<td>$J(x^i, z^i)$</td>
<td>$J(x(s), z(s \mid x))$</td>
</tr>
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<td>Aggregate</td>
<td>$z^i(x) := \frac{1}{N} \sum_{j=1}^{N} G_{ij} x^j$</td>
<td>$z(s \mid x) := \int_0^1 W(s, t)x(t)dt$</td>
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Remarks:

- Agents are a continuum in $[0, 1]$ (non-atomic)
- $W(s, t)$ represents the interaction among the non-atomic agents $s$ and $t$
- The agents cost function is the same in network and in graphon games
- A strategy profile is a function $x : [0, 1] \rightarrow \mathcal{X}$
Network versus graphon games

- Agents: \( i \in \{1, \ldots, N\} \)
- Interactions: \( G \in \mathbb{R}^{N \times N} \)
- Strategy: \( x^i \in X^i \)
- Cost function: \( J(x^i, z^i) \)
- Aggregate: \( z^i(x) := \frac{1}{N} \sum_{j=1}^{N} G_{ij} x^j \)

\[ z(s | x) := \int_0^1 W(s, t)x(t)dt \]

**Definition: Nash equilibrium**

A function \( \bar{x}(s) \in X(s) \) is a Nash equilibrium if for all \( s \in [0, 1] \)

\[ J(\bar{x}(s), z(s | \bar{x})) \leq J(\bar{x}, z(s | \bar{x})) \text{ for all } \bar{x} \in X(s) \]

Similarity with Wardrop equilibrium for non-atomic congestion games
[Wardrop, 1900], [Smith, 1979]
Network versus graphon games

- **Agents:** $i \in \{1, \ldots, N\}$
- **Interactions:** $G \in \mathbb{R}^{N \times N}$
- **Strategy:** $x^i \in \mathcal{X}^i$
- **Cost function:** $J(x^i, z^i)$
- **Aggregate:** $z^i(x) := \frac{1}{N} \sum_{j=1}^{N} G_{ij} x^j$

**Network games**

**Graphon games**

$s \in [0, 1]$  
$W : [0, 1]^2 \rightarrow [0, 1]$  
$x(s) \in \mathcal{X}(s)$  
$J(x(s), z(s \mid x))$  
$z(s \mid x) := \int_0^1 W(s, t)x(t)dt$

**Definition: Nash equilibrium**

A function $\bar{x}(s) \in \mathcal{X}(s)$ is a Nash equilibrium if for all $s \in [0, 1]$

$$J(\bar{x}(s), z(s \mid \bar{x})) \leq J(\tilde{x}, z(s \mid \bar{x}))$$ for all $\tilde{x} \in \mathcal{X}(s)$

Does a Nash equilibrium exist? Is it unique?
## The graphon operator

<table>
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<th>Adjacency matrix</th>
<th>Graphon operator</th>
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<tr>
<td>$G \in \mathbb{R}^{N \times N}$</td>
<td>$\mathbb{W} : L^2([0, 1]) \mapsto L^2([0, 1])$</td>
</tr>
<tr>
<td>$v \mapsto Gv$</td>
<td>$f \mapsto (\mathbb{W}f)(s) = \int_0^1 \mathbb{W}(s, t)f(t)dt.$</td>
</tr>
<tr>
<td>$Gv = \lambda v$</td>
<td>$(\mathbb{W}f)(s) = \lambda f(s)$</td>
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### Properties of $\mathbb{W}$ - [Lovász, 2012]

1. $\mathbb{W}$ is a linear, continuous, bounded operator;
2. all the eigenvalues of $\mathbb{W}$ are real;
3. $\|\mathbb{W}\| := \sup_{f \in L^2([0,1]), \|f\|_L = 1} \|\mathbb{W}f\|_L = \lambda_{\text{max}}(\mathbb{W}).$
An example

Consider an infinite population of agents which are spatially located along a line (e.g. a street).

- $s \in [0, 1] = \text{position along line}$
- influence between agents is a decreasing function of the spatial distance
- central agents are affected more

\[ W(s, t) = \min(s, t)(1 - \max(s, t)) \]

Spectral properties:

\[ \lambda_h := \frac{1}{\pi^2 h^2}, \quad \psi_h(s) := \sqrt{2} \sin(h\pi s) \quad \forall h \in \{1, 2, \ldots \infty\} \]

$W$ has an infinite, but countable, number of nonzero eigenvalues, with an accumulation point at zero. Moreover, $\lambda_{\max}(W) = \frac{1}{\pi^2}$.
Reformulation as fixed point of the best response mapping

\[ \bar{x}(s) = \arg \min_{\tilde{x} \in \mathcal{X}(s)} J(\tilde{x}, z(s \mid \bar{x})) \]

- consider a fixed strategy profile \( x(s) \)
- the corresponding local aggregate function is
  \[ z_x(s) := z(s \mid x) = \int_0^1 W(s, t)x(t)dt = (\mathbb{W}x)(s) \]

- Define the best response operator \( \mathbb{B} \)
  \[ (\mathbb{B}z)(s) := \arg \min_{\tilde{x} \in \mathcal{X}(s)} J(\tilde{x}, z(s)), \]

- the best response mapping is
  \[ x \mapsto \mathbb{B}z_x = \mathbb{B}Wx \]

**Lemma - Equivalent characterization**

\( \bar{x} \) is a Nash equilibrium iff it is a fixed point of the game operator \( \mathbb{B}W \), i.e.

\[ \bar{x} = \mathbb{B}W\bar{x}. \]
Existence and uniqueness

**Assumption on cost and strategy sets**
- \( J(x, z) \) is \( C^1 \) and strongly convex in \( x \) uniformly in \( z \) (constant \( \mu_J \))
  \[ \nabla_x J(x, z) \] is Lipschitz in \( z \) uniformly in \( x \) (constant \( \ell_J \))
- \( X(s) \) convex and closed \( \forall s \in [0, 1] \), \( \exists \mathcal{X} \) compact s.t. \( X(s) \subseteq \mathcal{X}, \forall s \).

**Proof idea:** prove that \( BW \) is a contraction

1. prove that \( B \) is Lipschitz

   \[
   \|Bz_1 - Bz_2\|_{L^2} \leq \frac{\ell_J}{\mu_J} \|z_1 - z_2\|_{L^2}
   \]

2. combining with \( W \) we get

   \[
   \|BWx_1 - BWx_2\|_{L^2} \leq \frac{\ell_J}{\mu_J} \|Wx_1 - Wx_2\|_{L^2} = \frac{\ell_J}{\mu_J} \|W(x_1 - x_2)\|_{L^2}
   \]

   \[
   \leq \frac{\ell_J}{\mu_J} \|W\| \|x_1 - x_2\|_{L^2} = \frac{\ell_J}{\mu_J} \lambda_{\max}(W) \|x_1 - x_2\|_{L^2}
   \]

3. apply Banach fixed point theorem

**Theorem**

\[
\frac{\ell_J}{\mu_J} \lambda_{\max}(W) < 1 \quad \Rightarrow \quad \text{existence and uniqueness}
\]
Linear quadratic graphon games

\[ J(x^i, z^i) = \frac{1}{2} (x^i)^2 - x^i [Kz^i + a] \]

• a Nash equilibrium exists and is unique if

\[ \frac{\ell_J}{\mu_J} \lambda_{\text{max}}(W) < 1 \iff K < \frac{1}{\lambda_{\text{max}}(W)} \]

→ compare with: [Ballester et al., 2006], [Jackson and Zenou, 2014]

• for game of strategic complements \((K > 0)\) equilibrium is proportional to Bonacich centrality

\[ \bar{x}(s) = a((I - K W)^{-1}1_{[0,1]})(s) \]

→ centrality measures for graphons:

[Avella-Medina, Parise, Schaub, Segarra. 2017]
Example - cont’d

\[ J(x^i, z^i) = \frac{1}{2} (x^i)^2 - x^i [Kz^i + a] \]

- Use minmax graphon and recall \( \lambda_{\text{max}}(\mathbb{W}) = \frac{1}{\pi^2} \)
- Set \( K = 0.5 \) (for uniqueness).
Comparative statics

How does the equilibrium change if the graphon changes from $W$ to $\tilde{W}$?

**Theorem**

Let $x_{\text{max}} := \max_{x \in X} \|x\|$. Then under the previous assumptions

$$\|\bar{x} - \tilde{x}\|_{L^2} \leq \frac{\ell_J/\mu_J x_{\text{max}}}{1 - \ell_J/\mu_J \lambda_{\text{max}}(W)} \left\|W - \tilde{W}\right\|$$

**Proof idea:**

$$\|\bar{x} - \tilde{x}\|_{L^2} = \|B\bar{W}\bar{x} - B\tilde{W}\tilde{x}\|_{L^2} \leq \frac{\ell_J}{\mu_J} \|\bar{W}\bar{x} - \tilde{W}\tilde{x}\|_{L^2}$$

$$\leq \frac{\ell_J}{\mu_J} \|\bar{W}\bar{x} - \bar{W}\tilde{x}\|_{L^2} + \frac{\ell_J}{\mu_J} \|\bar{W}\tilde{x} - \tilde{W}\tilde{x}\|_{L^2}$$

$$\leq \frac{\ell_J}{\mu_J} \|W\| \|\bar{x} - \tilde{x}\|_{L^2} + \frac{\ell_J}{\mu_J} \|W - \tilde{W}\| \|\tilde{x}\|_{L^2}$$
Step 2:
finite population
**Theorem: Nash equilibrium distance**

Suppose that \( W \) is Lipschitz continuous and fix any tolerance \( \delta \ll 1 \).

With probability at least \( 1 - 2\delta \)

\[
\| \bar{x}^{[N]} - \bar{x} \|_{L^2} \leq K \sqrt{\frac{\log(N/\delta)}{N}}
\]

**Proof idea:**

- map any finite network game to a graphon game with piece-wise constant graphon \( W^{[N]} \)

- relate equilibria distance to graphon operator distance:

\[
\| \bar{x}^{[N]} - \bar{x} \|_{L^2} \leq \tilde{K}_1 \| W^{[N]} - W \|
\]

- bound the graphon operator distance (improvement on [Lovász, 2012]):

\[
\| W^{[N]} - W \| \leq \tilde{K}_2 \sqrt{\frac{\log(N/\delta)}{N}}
\]
Use minmax graphon and recall $\lambda_{\text{max}}(\mathcal{W}) = \frac{1}{\pi^2}$. Set $\alpha = 0.5$ (for uniqueness).

- Plot the equilibrium in sampled network games for $N = 10, 50, 200, 2000$
- Plot expected distance over 100 realizations
Step 3:
Interventions
Welfare maximization in LQ network games

\[ J(x^i, z^i) = \frac{1}{2}(x^i)^2 - x^i[Kz^i + a^i], \quad K > 0 \]

This could model for example peer pressure in education: Calvó-Armengol, Patacchini, Zenou (2009)

- \(x^i\) = student effort
- \(K\) = level of peer pressure
- \(a^i\) = effort in isolation
Welfare maximization in LQ network games

\[ J(x^i, z^i | \beta^i) = \frac{1}{2}(x^i)^2 - x^i[Kz^i + a^i + \beta^i] \]

Per-capita welfare maximization problem - Galeotti et al., (2017):

\[
\begin{align*}
\max_{\beta \in \mathbb{R}^N} & \quad T^{[N]}_\beta := -\frac{1}{N} \sum_{i=1}^N J(\bar{x}^i, \bar{z}^i | \beta^i) \\
\text{s.t.} & \quad \sum_{i=1}^N (\beta^i)^2 \leq C^{[N]},
\end{align*}
\]

Network heuristic

\[ \beta^{[N]}_{nh} := \sqrt{C^{[N]}} v^{[N]}_1 \]

where \( v^{[N]}_1 \) is the dominant eigenvector of \( G^{[N]} \)

Graphon heuristic

\[ [\beta^{[N]}_{gh}]_i := \kappa^{[N]} \cdot \psi_1(s_i), \]

where \( \psi_1 \) is the dominant eigenfunction of \( W \)
Performance of the graphon heuristic

**Theorem**

If further $\lambda_1(W) > \lambda_2(W)$ and $C^{[N]} = O(N)$, with probability $1 - 2\delta$

$$|T_{nh}^{[N]} - T_{gh}^{[N]}| = O\left(\sqrt{\log(N/\delta)}\right).$$

**Proof idea:**

- i) $T^{[N]} = \frac{1}{2N} \|\bar{x}^{[N]}\|^2$ and ii) $\bar{x}^{[N]} = [I - \alpha \frac{G^{[N]}}{N}]^{-1} (a + \beta)$$

$$|T_{nh}^{[N]} - T_{gh}^{[N]}| \leq \frac{\sqrt{C^{[N]}} + a\sqrt{N}}{N} \frac{1}{(1 - \eta K \lambda_1(W))^2} \|\beta_{nh}^{[N]} - \beta_{gh}^{[N]}\|$$

$$\approx \frac{\sqrt{C^{[N]}} + a\sqrt{N}}{N} \sqrt{C^{[N]}} \frac{\sqrt{C^{[N]}}}{(1 - \eta K \lambda_1(W))^2} \|\varphi_1^{[N]} - \varphi_1\|_{L^2}$$

- By Davis-Kahan theorem

$$\|\varphi_1^{[N]} - \varphi_1\|_{L^2} \leq \sqrt{2} \frac{\|W^{[N]} - W\|}{\lambda_1(W) - \lambda_2(W)} = O\left(\sqrt{\log(N/\delta)}\right)$$
The community model

**SBM model**

- generalize to $K$ communities
- each agent belongs to community $k$ with probability $w_k$
- agents in community $l, k$ connect with probability $q_{l,k}$

(e.g. Currarini et al. (2009) community = students of different race)

**How to compute the dominant eigenfunction?**

- Let $D := \text{diag}([w_k]) \in \mathbb{R}^{4 \times 4}$
- Let $Q := [q_{l,k}] \in \mathbb{R}^{4 \times 4}$
- Let $\nu_1$ dominant eigenvector of $QD \in \mathbb{R}^{4 \times 4}$
- Then $\psi_1(s)$ is piece-wise constant with values given by $\nu_1$
The community model - cont’d

The graphon heuristic is much **simpler to implement**
Step 4: Incomplete information
An incomplete information sampled network game $G^{\text{in}}(X, J, W)$ is a game

- with a random number $N$ of agents
- with types $\{t^i\}_{i=1}^N$ sampled i.i.d from $\mathcal{U}[0, 1]$
- interacting according to a network $G^{[N]}$ sampled from the graphon $W$
- each agent $i$ has information about: $W$, $t^i$, $X$ and $J$
- while is uninformed about $G^{[N]}$ and the other agents types $t^{-i}$
Symmetric Bayesian Nash equilibrium

• Suppose agent of type $s$ play $b(s)$ (symmetric case)
• The expected cost of an agent of type $t^i = s$ playing $x(s) \in \mathcal{X}(s)$ is

$$J_{\text{exp}}(x(s) \mid b) = \mathbb{E}_{N,t-i, \text{links}} \left[ J \left( x(s), \frac{1}{N-1} \sum_{j \neq i} [G^N]_{ij} b(t^j) \right) \right]$$

**Symmetric Bayesian Nash equilibrium**

$b(s) \in \mathcal{X}(s)$ is a symmetric $\varepsilon$-Bayesian Nash equilibrium if for all $s \in [0, 1]$

$$J_{\text{exp}}(b(s) \mid b) \leq J_{\text{exp}}(\tilde{x} \mid b) + \varepsilon$$

for all $\tilde{x} \in \mathcal{X}(s)$.

Symmetric Bayesian Nash eq. are strictly related to graphon Nash eq.
Linear quadratic games

**Theorem**

\(\bar{x}\) is a Nash equilibrium of \(G(\bar{x}, J, W)\) iff it is a symmetric Bayesian Nash equilibrium of \(G^{in}(\bar{x}, J, W)\).

**Proof idea:**

- \(\bar{x}\) Graphon equilibrium iff \(J(\bar{x}(s), \bar{z}(s)) \leq J(\tilde{x}, \bar{z}(s)), \forall \bar{x}, s\)

\[\bar{z}(s) = \int_0^1 W(s, t)\bar{x}(t)dt\]

- Fix \(b = \bar{x}\), by linearity in aggregate: \(J_{\text{exp}}(x(s) \mid \bar{x}) = J(x(s), z_{\text{exp}}(s))\) where

\[z_{\text{exp}}(s) := \mathbb{E}_{N, t-i, \text{links}} \left[ \frac{1}{N-1} \sum_j [G^{[N]}]_{ij} \bar{x}(t^j) \right]\]

\(\bar{x}\) Bayesian equilibrium iff \(J(\bar{x}(s), z_{\text{exp}}(s)) \leq J(\tilde{x}, z_{\text{exp}}(s)), \forall \bar{x}, s.\)

- Conclusion follows from \(z_{\text{exp}}(s) = \bar{z}(s)\)

\[z_{\text{exp}}(s) = \mathbb{E}_N \mathbb{E}_{t-i \mid N} \mathbb{E}_{\text{links} \mid t-i \mid N} \left[ \frac{1}{N-1} \sum_{j \neq i} [G^{[N]}]_{ij} \bar{x}(t^j) \right] = \mathbb{E}_N \frac{1}{N-1} \sum_{j \neq i} \mathbb{E}_{t} \left[ W(s, t^j) \bar{x}(t^j) \right] = \mathbb{E}_N \frac{1}{N-1} \sum_{j \neq i} \mathbb{E}_{t} \left[ W(s, t^j) \bar{x}(t^j) \right] = \mathbb{E}_N \frac{1}{N-1} \sum_{j \neq i} \bar{z}(s) = \mathbb{E}_N \bar{z}(s) = \bar{z}(s)\]
Theorem

Further suppose that

- $J(x, z)$ is Lipschitz continuous in $z$ uniformly over $x$
- agents know that $N \geq N_{\text{min}}$

The Nash equilibrium of $G(x, J, W)$ is a symmetric $\varepsilon$-Bayesian Nash equilibrium with

$$\varepsilon = O \left( \sqrt{\frac{\log(N_{\text{min}})}{N_{\text{min}}}} \right).$$

Proof idea:

- For general cost

$$J_{\text{exp}}(x(s) \mid \bar{x}) = E_{N, t-i, \text{links}} \left[ J(x(s), \frac{1}{N-1} \sum_j [G^N]_{ij}(t^j)) \right] \neq J(x(s), z_{\text{exp}}(s))$$

$$= E_{\zeta_{\bar{x}}(s)} [J(x(s), \zeta_{\bar{x}}(s))]$$

- Prove that $\zeta_{\bar{x}}(s)$ concentrates around $z_{\text{exp}}(s)$ for $N$ large
- Use Lipschitz condition to show that

$$J_{\text{exp}}(x(s) \mid \bar{x}) \approx J(x(s), z_{\text{exp}}(s)) = J(x(s), \bar{z}(s))$$
Summary

- Define graphon games and study equilibrium properties
- Graphon equilibrium is a good approximation for sampled network games
- Shown how to design robust interventions using graphon model
- Foundation for graphon equilibrium in incomplete information games