A model for multilane traffic flow on simple networks

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Resilient Control of Infrastructure Networks
September 24–27, 2019 – Turin, Italy
LWR macroscopic traffic flow model

\[ \begin{align*}
    \partial_t \rho + \partial_x \rho \nu(\rho) &= 0 \quad (t, x) \in [0, T] \times \mathbb{R} \\
    \rho(0, x) &= \rho_o(x) \\
    \rho(t, x) &= \text{density of vehicles} \in [0, R] \\
    \nu(\rho) &= \text{speed law (density dependent)}
\end{align*} \]

[Lighthill-Whitham, 1955; Richards, 1956]
LWR macroscopic traffic flow model

\[
\begin{aligned}
\partial_t \rho + \partial_x \rho \, v(\rho) &= 0 \quad (t, x) \in [0, T] \times \mathbb{R} \\
\rho(0, x) &= \rho_o(x) \\
\end{aligned}
\]

\(\rho(t, x)\) density of vehicles \(\in [0, R]\)

\(v(\rho)\) speed law (density dependent)

\[v \geq 0, \; v' < 0, \; v(R) = 0\]

[Lighthill-Whitham, 1955; Richards, 1956]
LWR macroscopic traffic flow model

\[
\begin{aligned}
\frac{\partial t}{\partial t} \rho + \frac{\partial x}{\partial x} \rho \, v(\rho) &= 0 \quad (t, x) \in [0, T] \times \mathbb{R} \\
\rho(0, x) &= \rho_0(x) \quad x \in \mathbb{R}
\end{aligned}
\]

\(\rho(t, x)\) density of vehicles \(\in [0, R]\)

\(v(\rho)\) speed law (density dependent)

\(v \geq 0, \ v' < 0, \ v(R) = 0\)

\[v(\rho) = V \left(1 - \frac{\rho}{R}\right)\]

[Lighthill-Whitham, 1955; Richards, 1956]
A multilane model: two lanes

\[
\begin{align*}
\frac{\partial_t \rho_1 + \partial_x \rho_1 v_1(\rho_1)}{\rho_1(0, x)} &= \rho_{o,1}(x) \\
\frac{\partial_t \rho_2 + \partial_x \rho_2 v_2(\rho_2)}{\rho_2(0, x)} &= \rho_{o,2}(x)
\end{align*}
\]

\[
S(\rho_1, \rho_2) = (v_2(\rho_2) - v_1(\rho_1)) \begin{cases} 
\rho_1 & v_2(\rho_2) \geq v_1(\rho_1) \\
\rho_2 & v_2(\rho_2) < v_1(\rho_1)
\end{cases}
\]

[Holden-Risebro, 2019]
Extending the multilane model

- The number of lanes can change
- The speed laws can change

**Example 1:** 1-to-1 junction, from 2 to 3 lanes

Extending the multilane model

- The number of lanes can change
- The speed laws can change

Example 2: 1-to-1 junction, from 3 to 2 lanes

\[ x = 0 \]

Extending the multilane model

- The number of lanes can change
- The speed laws can change

Example 3: 2-to-1 junction, from 1+2 to 2 lanes

The model

\begin{align*}
\text{lane 2} & \quad \text{lane 3} \\
\text{lane 1} & \quad \text{lane 2} \\
& \quad \text{lane 1}
\end{align*}

$x = 0$

\begin{align*}
\text{lane 3} & \quad \text{lane 2} \\
\text{lane 1} & \quad \text{lane 2} \\
& \quad \text{lane 1}
\end{align*}

$x = 0$
The model

Same number of lanes $M$ on the left and on the right of $x = 0$
The model

- Same number of lanes $M$ on the left and on the right of $x = 0$
- No lane change from *active* to fictive lane
The model

- Same number of lanes $M$ on the left and on the right of $x = 0$
  - No lane change from active to fictive lane
  - Adjust the initial data: $\rho_{o,j} : \mathbb{R} \to [0, 1]$ for $j = 1, \ldots, M$ with
    
    \[
    \rho_{o,j}(x) = 0 \quad \text{for } x \in ]-\infty, 0[ \text{ and } j \notin M_\ell, \\
    \rho_{o,j}(x) = 1 \quad \text{for } x \in ]0, +\infty[ \text{ and } j \notin M_r.
    \]
The model

For $j = 1, \ldots, M$, each equation of the system reads

$$\partial_t \rho_j + \partial_x \rho_j v_j(x, \rho_j) = S_{j-1}(x, \rho_{j-1}, \rho_j) - S_j(x, \rho_j, \rho_{j+1})$$
The model

For $j = 1, \ldots, M$, each equation of the system reads

$$\partial_t \rho_j + \partial_x \rho_j v_j(x, \rho_j) = S_{j-1}(x, \rho_{j-1}, \rho_j) - S_j(x, \rho_j, \rho_{j+1})$$

with

$$v_j(x, u) = H(x) v_{r,j}(u) + (1 - H(x)) v_{\ell,j}(u) \quad \text{for } j = 1, \ldots, M,$$

$$S_j(x, u, w) = H(x) S_{r,j}(u, w) + (1 - H(x)) S_{\ell,j}(u, w) \quad \text{for } j = 1, \ldots, M - 1,$$

$$S_0 = S_M = 0$$
1-to-1 junction: from two to three lanes

\[
\begin{align*}
\partial_t \rho_1 + \partial_x \rho_1 v_1(x, \rho_1) &= -S_1(x, \rho_1, \rho_2) \\
\partial_t \rho_2 + \partial_x \rho_2 v_2(x, \rho_2) &= S_1(x, \rho_1, \rho_2) - S_{r,2}(\rho_2, \rho_3) \\
\partial_t \rho_3 + \partial_x \rho_3 v_3(x, \rho_3) &= S_{r,2}(\rho_2, \rho_3)
\end{align*}
\]

We impose \( \rho_{o,3}(x) = 0 \) for \( x < 0 \) and \( S_{\ell,2}(u, w) = 0 \).
1-to-1 junction: from two to three lanes

\[ \rho_{o,1}(x) = 0.7, \quad \rho_{o,2}(x) = 0.6, \quad \rho_{o,3}(x) = 0.5 \chi_{[0, +\infty]}(x). \]

\[ V_\ell = 1.5, \quad V_r = 1 \]

\[ V_\ell = 1.5, \quad V_r = 2 \]
1-to-1 junction: from three to two lanes

\[
\begin{align*}
\partial_t \rho_1 + \partial_x \rho_1 v_1(x, \rho_1) &= -S_1(x, \rho_1, \rho_2) \\
\partial_t \rho_2 + \partial_x \rho_2 v_2(x, \rho_2) &= S_1(x, \rho_1, \rho_2) - S_{\ell,2}(\rho_2, \rho_3) \\
\partial_t \rho_3 + \partial_x \rho_3 v_3(x, \rho_3) &= S_{\ell,2}(\rho_2, \rho_3)
\end{align*}
\]

We impose \( \rho_{o,3}(x) = 1 \) for \( x > 0 \) and \( S_{r,2}(u, w) = 0 \).
1-to-1 junction: from three to two lanes

\[ \rho_{o,1}(x) = 0.7, \quad \rho_{o,2}(x) = 0.6, \quad \rho_{o,3}(x) = 0.5 \chi_{]-\infty,0]}(x) + 1 \chi_{]0,\infty[}(x). \]

\[ V_\ell = 1.5, \quad V_r = 1 \]

\[ V_\ell = 1.5, \quad V_r = 2 \]
2-to-1 junction: from one+two to two lanes

\[ \begin{align*}
\frac{\partial}{\partial t} \rho_1 + \frac{\partial}{\partial x} \rho_1 v_1(x, \rho_1) &= -S_{r,1}(\rho_1, \rho_2) \\
\frac{\partial}{\partial t} \rho_2 + \frac{\partial}{\partial x} \rho_2 v_2(x, \rho_2) &= S_{r,1}(\rho_1, \rho_2) - S_{\ell,2}(\rho_2, \rho_3) \\
\frac{\partial}{\partial t} \rho_3 + \frac{\partial}{\partial x} \rho_3 v_3(x, \rho_3) &= S_{\ell,2}(\rho_2, \rho_3)
\end{align*} \]

We impose \( \rho_{o,3}(x) = 1 \) for \( x > 0 \), \( S_{r,2}(u, w) = 0 \) and \( S_{\ell,1}(u, w) = 0 \).
2-to-1 junction: from one+two to two lanes

\[ \rho_{o,1}(x) = 0.4, \quad \rho_{o,2}(x) = 0.6, \quad \rho_{o,3}(x) = 0.5 \chi_{-\infty,0}(x) + 1 \chi_{0,\infty}(x). \]

\[ V_\ell = 1.5, \quad V_r = 1 \]

\[ V_\ell = V_r = 1.5 \]

\[ V_\ell = 1.5, \quad V_r = 2 \]
Main Publications

- H. Holden, N.H. Risebro,
  *Models for Dense Multilane Vehicular Traffic*,

- P. Goatin, E. Rossi,
  *A multilane macroscopic traffic flow model for simple networks*,