How to Achieve Fast Spread in Controlled Evolutionary Dynamics

Lorenzo Zino
(joint work with G. Como and F. Fagnani)

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Evolutionary dynamics on graphs

Genetically modified mosquitoes to help control dengue, malaria
Our goals

Define a novel framework to model evolutionary dynamics, which allows for including control

Understand how the spreading time is influenced by the graph topology and the control policy

Design a feedback control policy to speed up the spreading process and test it on a real-world case study
Weighted graph

- **Connected graph**
- **Node set** $\mathcal{V} = \{1, \ldots, n\}$
- **Undirected links**, symmetric **weight matrix** $\mathcal{W} \in \mathbb{R}_{\geq 0}^{n \times n}$
Evolutionary dynamics

- \( X_i(t) \in \{0, 1\} \) state of node \( i \) at time \( t \in \mathbb{R}_{\geq 0} \):
  \[
  X_i(t) = \begin{cases} 
  1 & \text{if } i \text{ has the novel state at time } t \\
  0 & \text{if } i \text{ has the old state at time } t 
  \end{cases}
  \]

- Link \( \{i, j\} \) is activated by a \textbf{Poisson clocks} with rate \( W_{ij} \)

- If \( X_i(t) \neq X_j(t) \) \( \implies \) \text{conflict}: novel state wins w.p. \( \beta > 1/2 \)
External control

- A **target node** $m(t) \subset \mathcal{V}$ is selected\(^1\)
- Novel state is **introduced** in $m(t)$ with rate $u(t) \geq 0$
- Simplest choice: **constant control** $m(t) = m$, $u(t) = u$.
- **Feedback control** $m(t) = m(X(t))$, $u(t) = u(X(t))$

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\(^1\)We can generalize it to a target set of nodes
Markov jump process

\[ \lambda_i^+(x) = (1 - x_i) \left[ \beta \sum_j W_{ij} x_j + u(t) \delta_{i=m(t)} \right] \]

\[ \lambda_i^-(x) = x_i(1 - \beta) \sum_j W_{ij} (1 - x_j) \]

\[ \lambda_i^+(x) \]

\[ \lambda_i^-(x) \]

\[ X = \mathbb{1} \text{ unique absorbing state} \implies \text{novel state will spread} \]

\[ \tau = \mathbb{E} \left[ \inf t : X(t) = \mathbb{1} \right] \]

\[ \nu = \mathbb{E} \left[ \int_0^\infty u(t) dt \right] \]

\[ X(t) \textbf{ Markov jump process} \text{ with } X(0) = 0 \mathbb{1}. \]
Three observables

- $A(t) = \sum_i X_i(t)$ number of nodes with the novel state
- $B(t) = \sum_i \sum_j X_i(t)(1 - X_j(t))W_{ij}$ boundary between the two states
- $C(t) = (1 - X_{m(t)}(t))u(t)$ effective control in nodes with state 0
General results

Performance guarantees (PG)

If $B(t) + C(t) \geq f(A(t)) > 0$ for any $t \geq 0$, then

$$
\tau \leq \frac{\beta}{(2\beta - 1)f(0)} + \frac{1}{2\beta - 1} \sum_{a=1}^{n-1} \frac{1}{f(a)}
$$

Fundamental limitation (FL)

Called $T_h$ and $J_h$ the contributions to $\tau$ and $\nu$ each time $A(t) = h$, it holds

$$
\mathbb{E}[T_h] \geq \frac{1 - \mathbb{E}[J_h]}{B(t)}.
$$
Constant Control
**Constant control**

**Upper bound on the expected spreading time (UB)**

Let $\phi(a) : 1, \ldots, n - 1 \rightarrow \mathbb{R}$ be the minimum conductance. Then,

$$
\tau \leq \frac{\beta}{(2\beta - 1)u} + \frac{1}{2\beta - 1} \sum_{a=1}^{n-1} \frac{1}{\phi(a)}
$$

**Lower bounds on the expected spreading time (LB)**

I: Let $\eta(a) : 1, \ldots, n - 1 \rightarrow \mathbb{R}$ be the maximum expansiveness. Then,

$$
\tau \geq \frac{1}{u} + \sum_{a=1}^{n-1} \frac{1}{\eta(a)}
$$

II: Let $\xi$ be the (weighted) bottleneck of the graph. Then, $\tau \geq \xi^{-1}$.
Example 1: fast spread on expander graphs

Complete

Erdős-Rényi
Example I: fast spread on expander graphs

\[ \text{UB} + \text{LB} \implies \text{fast spread: } \tau = \Theta(\ln n) \]
Example II: slow spread on stochastic block models
Example II: slow spread on stochastic block models

- The fundamental limit allows fast spread
- Constant control: $UB + LB \iff$ slow spread: $\tau = \Theta(n)$
Example III: slow spread on rings
Example III: slow spread on rings

FL $\implies$ slow spread for any control policy

$$\tau \geq \frac{n - \nu}{2\nu} \in \Theta(n)$$

\begin{align*}
\beta = 0.7 & \quad & \text{time} \\
\beta = 0.8 & \quad & \text{time}
\end{align*}
To sum up...

😊 If the topology ensures fast spread with constant control...

...we reached our goal!

😢 If the fundamental limitation does not allow fast spread...

...there is no solution!

⇒ If the fundamental limitation allows fast spread, but constant control fails...

...we need to improve the control!
Feedback Control
Feedback control policy

- **Avoid waste**: \( m(t) \) moved in a random node with state 0
  \[ \Rightarrow \] no optimization on \( m(t) \) (라도 굵이 사용가능!) 

- **Contrast slowdowns**: velocity of the process proportional to \( B(t) \)
  \[ \Rightarrow \] \( u(t) \) should compensate when \( B(t) \) is small

\[
u(t) = u(A(t), B(t)) = \begin{cases} 
C - B(t) & \text{if } A(t) \neq n, \ B(t) < C \\
0 & \text{else}
\end{cases}
\]

**Upper bounds on spreading time and cost under feedback control**

\[
\tau \leq \frac{\beta}{(2\beta - 1)}C + \frac{1}{2\beta - 1} \sum_{a=1}^{n-1} \frac{1}{\max\{\phi(a), C\}}
\]

\[
\nu \leq \frac{\beta}{(2\beta - 1)} + \frac{1}{2\beta - 1} |\{a : \phi(a) < C\}|
\]
Application of feedback control policy to SBMs

- $C < \frac{1}{2} - \frac{1}{n}$ $\implies$ control activates only at bottleneck
- Upper bounds $\implies$ fast spread: $\tau \in \Theta(\ln n)$, $\nu \in \Theta(1)$

**Constant control policy**

**Feedback control policy**
Easy to control

Feedback controllable

Hard to control

Constant vs Feedback
FDA approves Zika-fighting genetically modified mosquitoes to help control dengue, malaria

By Sandee LaMotte, CNN
Updated 2126 GMT (0526 HKT) August 5, 2016

U.S. One Step Closer to Releasing Engineered Mosquito to Fight Zika
Community support for a field trial, required before any releases, is not guaranteed

Cases of Dengue Drop 91 Percent Due to Genetically Modified Mosquitoes
Case study: Zika in Rwanda

Zika Alert in Rwanda since 2016 [CDC, accessed online September 26, 2019]
Case study: model parameters

Location connected within a certain distance. Threshold set to 11.7 km: max distance traveled by mosquitoes to lay eggs [Bogojevic et al., J.Amer.Mosq.Cont.Ass., 2007]

Activation rate $w = 1/10$. 10 days life-cycle of *Aedes aegypti* [CDC Centers for Disease Control and Prevention, accessed online September 26, 2019]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Meaning</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>Number of locations</td>
<td>1621</td>
</tr>
<tr>
<td>$W_{ij}$</td>
<td>Activation rate</td>
<td>0.1</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Evolutionary advantage</td>
<td>0.53</td>
</tr>
<tr>
<td>$u$</td>
<td>Control rate (constant)</td>
<td>2</td>
</tr>
<tr>
<td>$C$</td>
<td>Control parameter (feedback)</td>
<td>1.5</td>
</tr>
<tr>
<td>$t$</td>
<td>Time unit</td>
<td>day</td>
</tr>
</tbody>
</table>
Case study: results of numerical simulations

 bois Same cost, performance improved: $\tau \downarrow -56\%, p\text{-value} << 0.001$
Conclusions and future works

Analytical tractable model for controlled evolutionary dynamics

- **General results** to bound spreading time and control cost
- For some networks, **constant control** guarantees fast spread
- **Feedback control** can strongly improve the performance

Current/future work

- Look for an **optimal control** strategy
- Use our tools to tackle **different problems** (e.g., slow the spread)

More details can be found in...

- *Controlling Evolutionary Dynamics in Networks: A Case Study*, IFAC Papers OnLine 51-23, pp. 349–354, 2018
- *Fast Spread in Controlled Evolutionary Dynamics*, Working Paper
Thank you for your attention!

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