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On a Centrality Maximization Game

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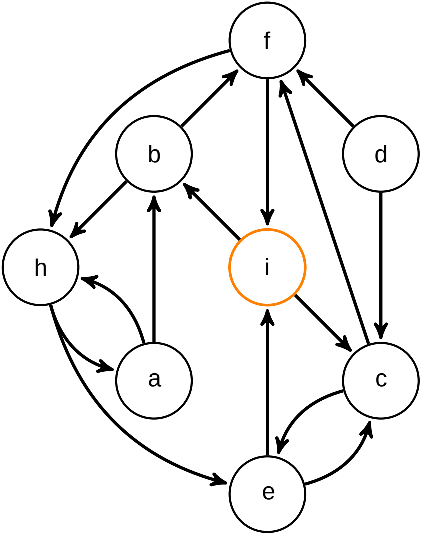
Workshop on Network Dynamics in the Social, Economic, and Financial Sciences

Bonacich Centrality



How to measure the importance $_i$ of a position?

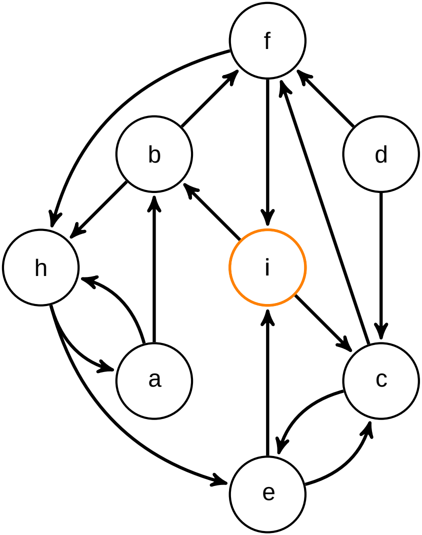
$$= \quad P^0 \quad + (1 \qquad) \; ;$$

where $P \quad _i \quad _i = 1$.

Bonacich centrality

Applications: visibility of a webpage on the internet.

Centrality Maximization Game

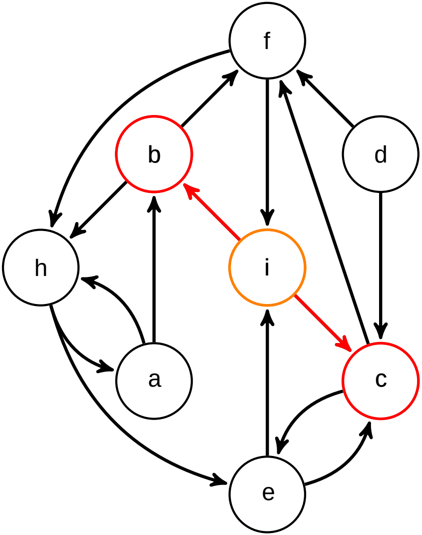


$(V; \; ; \; ; \; m) = (V; fA_i g_{i2V}; f u_i g_{i2V})$

$V =$ f players g

x) G (x).

Centrality Maximization Game



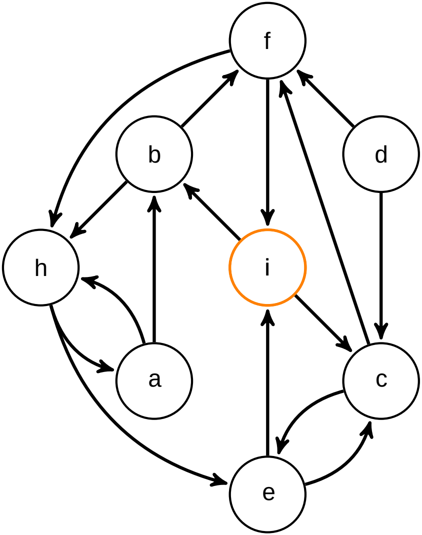
$(V;\ ;\ ;\ m) = (V; \{A_i\}_{i \in V}; \{u_i\}_{i \in V})$

$V = \{\text{players}\}$

action $x_i \in A_i$ of node i is the set of m nodes node i points to.

$x\ )\ G\ (x)$.

Centrality Maximization Game



$(V; \;; \;; m) = (V; \{A_i\}_{i \in V}; \{u_i\}_{i \in V})$

- $V = \{\text{players}\}$
- action $x_i \in A_i$ of node i is the set of m nodes node i points to.
- utility u_i of node i is the Bonacich centrality.

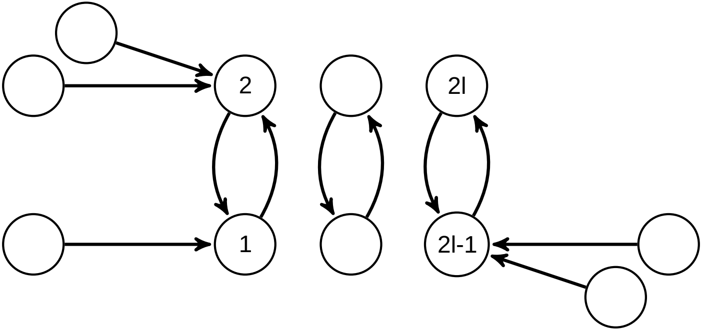
$x \Rightarrow G(x)$.

Results for m=1: Nash equilibria

x^* is a Nash equilibrium $\Leftrightarrow$ $G(x^*)$ is of type $C_2^{l;r}$, where $2l + r = |V|$:

- l number of 2-cliques;
- r nodes with in-degree equal to zero pointing at a 2-clique.

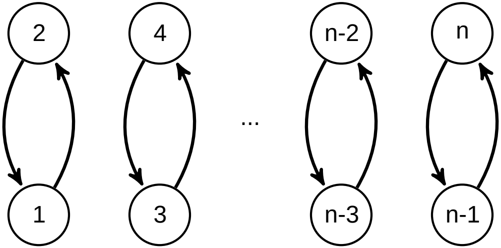
Example:



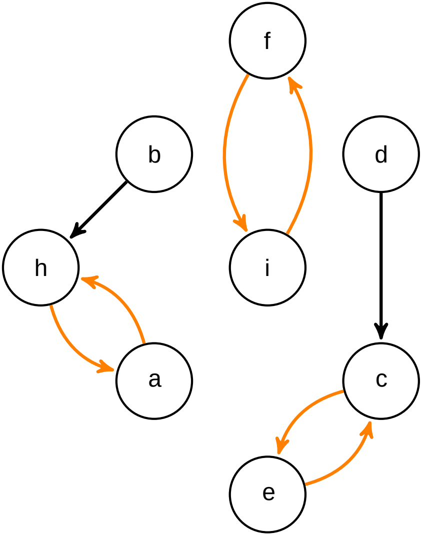
Results for m=1: strict Nash equilibria

If n is even, x is a strict Nash equilibrium, $G(x)$ is of type $C_2^{n=2;0}$.

If n is odd, there are no strict Nash equilibria.



Results for m=1: ordinal potential



Function : A ! N counting the number of 2-cliques is an ordinal potential ;

= 3

Results for m=1: Best Response Dynamics

The best response dynamics of V(; ; 1) always converges to a Nash equilibrium.

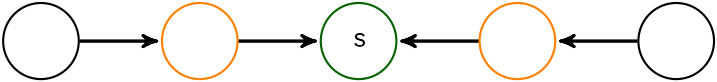
The limit set N is:

- set of $C_2^{n=2;0}$, if n is even;
- set of $C_2^{(n\ \ 1)=2;1}$, if n is odd;

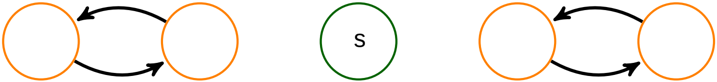
Idea of the proofs

Best Response Function

if $N_s^-(x) \neq \emptyset \Rightarrow$ $B_s(x_{-s}) = \{j \mid j \in N_s^-(x)\}$



if $N_s^-(x) = \emptyset \Rightarrow$ $B_s(x_{-s}) = \{j \mid j \in V\}$.

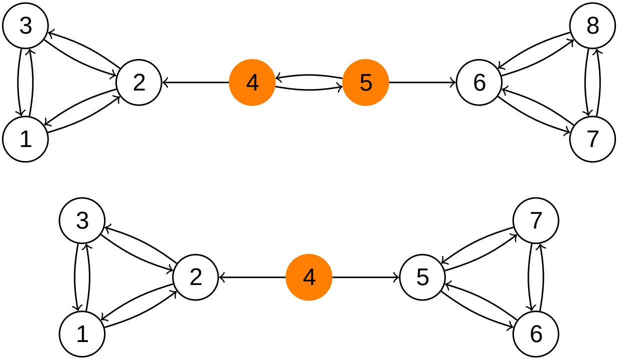


where $N_s^-(x)$ set of in-neighbors of s.

Results for m=2: Nash equilibria

If x is a Nash equilibrium:

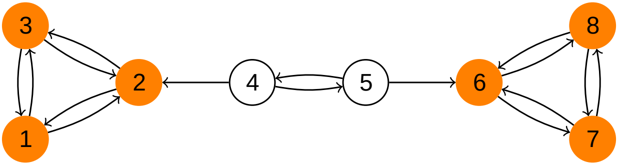
- every connected component of $G(x)$ is either a sink or a source. (if not isolated)
- every source component is a singleton or a 2-clique

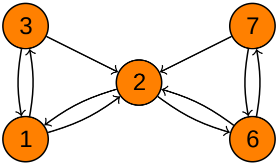


Results for m=2: Nash equilibria

If x is a Nash equilibrium:

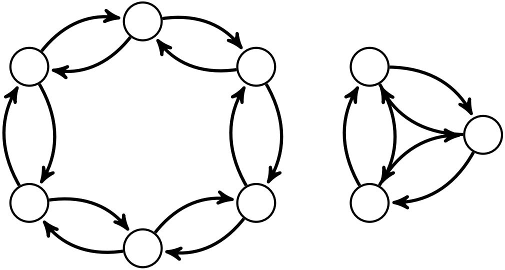
- every connected component of G(x) is either a sink or a source. (if not isolated)
- every source component is a singleton or a 2-clique
- every sink component is either a cycle or the Butter y graph





Results for m=2: Strict Nash equilibria

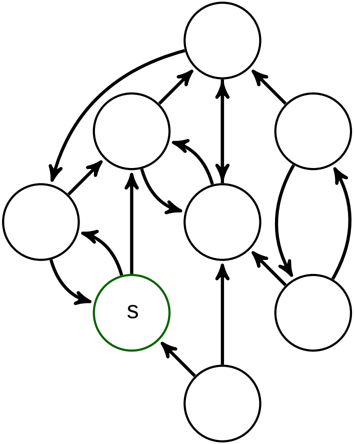
x is a strict Nash equilibrium, G(x) is undirected.



Idea of the proofs

Characteristics of the best response set

if $|N_s{}^{2}(x)| \quad 2$



where $N_s{}^{2}(x) = \{ j \mid dist_x(j;s) \quad 2\}$.

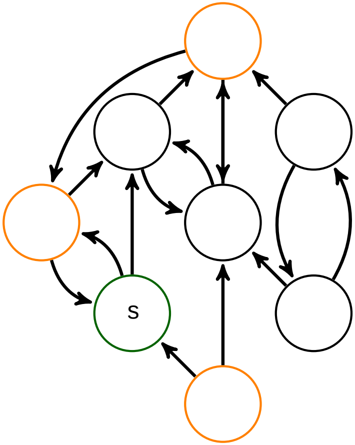
Idea of the proofs

Characteristics of the best response set

if $jN_s\ ^2(x)j \quad 2$

+

$B_s(x\ \ _s) \quad N_s\ ^2(x)$



where $N_s\ ^2(x) =\ f\ j\ j\ \mathrm{dist}_x(j; s) \quad 2g.$

Conclusions and Next Steps

Results:

The best strategy to maximize the Bonacich centrality is to act locally.

Strict Nash equilibria are all and only undirected graphs if $m = 1$ or $m = 2$.

Existence of an ordinal potential for $m = 1$.

Next steps:

Generalization to any m.