Non-local conservation laws for traffic flow modeling

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joint works with
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Outline of the talk

1. Introduction
2. Non-local conservation laws
   - Pre-existing non-local traffic flow models
3. A scalar traffic flow model with non-local velocity
   - Well-posedness
   - Limit model and numerical tests
4. A multi-class traffic flow model with non-local velocity
   - Existence of weak solutions
   - Numerical tests mixed traffic

Traffic flow modeling approaches

Macroscopic

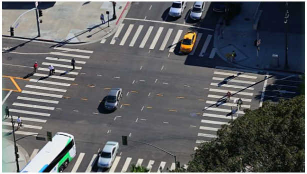
- collective behavior
- PDEs
- high densities

Microscopic

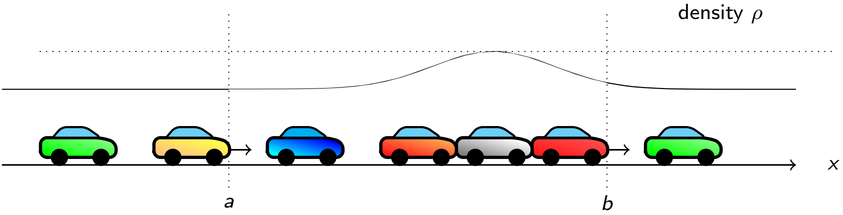
- individual vehicles and interactions
- ODE systems
- low and high densities

Mesoscopic

- gas-kinetic equations
- stochastic interactions



Conservation laws for traffic flow



$$\int_a^b \rho(t,x)dx = \text{total number of cars at time } t \text{ within the interval } [a,b]$$

$$\frac{d}{dt}\int_a^b \rho(t,x)dx = \text{(flux of cars entering at a)-(flux of cars exiting at b)}$$

$$= f(t,a) - f(t,b)$$

Mass conservation equation: $\partial_t \rho(t,x) + \partial_x f(\rho(t,x)) = 0$
Lighthill-Whitham-Richards model (1955-1956): $f(\rho(t,x)) = \rho(t,x)v(\rho(t,x))$

Non-local conservation laws

(Systems of) equations of the form

$$\partial_t U + \mathrm{div}_{\mathbf{x}} F(t, \mathbf{x}, U, w * U) = 0$$

with $t \in \mathbb{R}^+$, $\mathbf{x} \in \mathbb{R}^d$, $U(t, \mathbf{x}) \in \mathbb{R}^N$, $w(t, \mathbf{x}) \in \mathbb{R}^{m \times N}$

Applications

- sedimentation [Betancourt&al, Nonlinearity 2011]
- granular flows [Amadori-Shen, JHDE 2012]
- crowd dynamics [Colombo&al, ESAIM COCV 2011; AMS 2011; M3AS 2012]
- supply chains [Colombo-Herty-Mercier, ESAIM COCV 2011]
- conveyor belts [Göttlich&al, Appl. Math. Modell., 2014]
- gradient constraint [Amorim, Bull. Braz. Math. Soc., 2012]

- **Non-local models for pedestrian traffic**[1]
  - $$\partial_t \rho + div(\rho v(\rho)(\nu(x) + \mathcal{I}(\rho)) = 0,$$
    $$\mathcal{I}(\rho) = -\varepsilon \frac{\nabla(\rho * \omega)}{\sqrt{1 + \|\nabla(\rho * \omega)\|^2}}.$$
  - $$\partial_t \rho + div(\rho v(\rho * \omega)\nu(x)) = 0.$$
- **The Arrhenius model for vehicular traffic flow**[2]
  $$\partial_t \rho + \partial_x(\rho(1-\rho)\exp(-\omega_\eta * \rho)) = 0,$$
  $$\omega_\eta * \rho(t,x) = \int_x^{x+\eta} \frac{J_0}{\eta}\rho(t,y)dy.$$
- **The Blandin-Goatin model for vehicular traffic flow (2016)**
  $$\partial_t \rho + \partial_x \left(\rho v \left(\int_x^{x+\eta} \rho(t,y)\omega_\eta(y-x)dy\right)\right) = 0.$$
- **The Friedrich-Kolb-Göttlich model for vehicular traffic flow (2018)**
  $$\partial_t \rho + \partial_x \left(g(\rho)(\omega_\eta * v(\rho))\right) = 0.$$

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[1] [Colombo-Garavello-Mercier,2012],[Colombo-Herty-Mercier, 2011]

[2] [Sopasakis-Katsoulakis, SIAM J. APPL. Math., 2006]

A model with non-local velocity[3]

Traffic model with downstream non-local velocity

$$\partial_t \rho(t,x) + \partial_x \left(f(\rho(t,x)) V(t,x) \right) = 0$$

where

$$V(t,x) = v\left(\int_x^{x+\eta} \rho(t,y)\omega_\eta(y-x)\, dy \right), \quad \eta > 0$$

(H)

$$\begin{array}{ll} f \in \mathbf{C}^1(I;\mathbb{R}^+), & I = [a,b] \subseteq \mathbb{R}^+, \\ v \in \mathbf{C}^2(I;\mathbb{R}^+) \text{ s.t. } & v' \leq 0, \\ \omega_\eta \in \mathbf{C}^1([0,\eta];\mathbb{R}^+) \text{ s.t. } & \omega_\eta' \leq 0 \text{ and } \int_0^\eta \omega_\eta(x)dx := J_0,\ \forall \eta > 0,\ \lim_{\eta\to\infty} \omega_\eta(0) = 0. \end{array}$$

- traffic model: $F(\rho, \rho * \omega) = \rho V(\rho * \omega)$,
  [Blandin-Goatin, Numerische Mathematik, 2016]
- Arrhenius look-ahead dynamics: $F(\rho, \rho * \omega) = \rho(1-\rho)e^{-(\rho * \omega)}$
  [Sopasakis-Katsoulakis, SIAM 2006] [Kurganov-Polizzi, NHM 2009] [Li-Li, NHM 2011]

[3] [Chiarello-Goatin, ESAIM M2AN 2018]

Finite acceleration

The model avoids the infinite acceleration drawback of classical macroscopic models:

$$\begin{aligned} & \dot{x}(t) = V(t, x(t)), && t > 0 \\ \Longrightarrow \quad & \ddot{x}(t) = V_t(t, x(t)) + V(t, x(t)) V_x(t, x(t)), && t > 0 \end{aligned}$$

If $\rho(t, \cdot) \in \mathbf{L^1} \cap \mathbf{L}^\infty$, we have

$$\begin{aligned} \|V_t\|_\infty &= 2\omega_\eta(0)\|v\|_\infty \left\|v'\right\|_\infty \|f\|_\infty \\ \|V_x\|_\infty &= 2\omega_\eta(0)\left\|v'\right\|_\infty \|\rho\|_\infty \end{aligned}$$

Well-posedness

Theorem

[Blandin-Goatin, NumMath 2016; Goatin-Scialanga, NHM 2016; Chiarello-Goatin, ESAIM M2AN 2018]

Let $\rho_0 \in \mathrm{BV}(\mathbb{R}; I)$. Then the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x \left(f(\rho) V(t,x) \right) = 0 & x \in \mathbb{R}, t > 0 \\ \rho(0,x) = \rho_0(x) & x \in \mathbb{R} \end{cases}$$

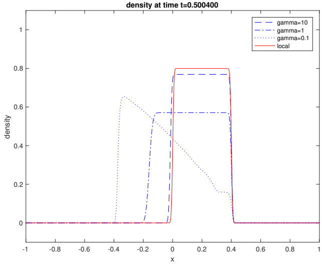
admits a unique weak entropy solution ($\rho^\eta \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathrm{BV}$), such that

$$\min_{\mathbb{R}}\{\rho_0\} \leq \rho^\eta(t,x) \leq \max_{\mathbb{R}}\{\rho_0\} \quad \text{for a.e. } x \in \mathbb{R},\ t > 0$$

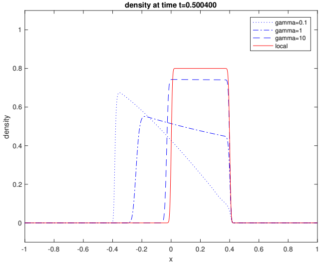
LIMIT $\eta \nearrow +\infty$ [4]

$$\partial_t \rho + \partial_x \left(\rho v(\rho * \omega_\eta)\right) = 0 \quad \rightarrow \quad \partial_t \rho + \partial_x \rho = 0$$

We consider $v(\rho) = 1 - \rho$ and $\rho_0(x) = \begin{cases} 0.8 & \text{if } -0.5 < x < -0.1 \\ 0 & \text{otherwise} \end{cases}$



$\omega_\eta = 1/\eta$



$\omega_\eta = 2(\eta - x)/\eta^2$

[4] [Chiarello-Goatin, ESAIM M2AN 2018]

A multi-class model with non-local velocity[5]

Multi-class traffic model with downstream non-local velocity

$$\partial_t \rho_i(t,x) + \partial_x \left(\rho_i(t,x) v_i((r * \omega_i)(t,x))\right) = 0, \qquad i = 1, ..., M,$$

where

$$r(t,x) := \sum_{i=1}^{M} \rho_i(t,x), \quad v_i(\xi) := v_i^{\mathbf{max}} \psi(\xi),$$

$$(r * \omega_i)(t,x) := \int_x^{x+\eta_i} r(t,y)\omega_i(y-x)\,\mathrm{d}y\,,$$

(H)

$$\omega_i \in \mathbf{C}^1([0,\eta_i];\mathbb{R}^+), \qquad \omega_i' \le 0, \quad \int_0^{\eta_i} \omega_i(y)\,\mathrm{d}y = J_i.$$

$$W_0 := \max_{i=1,\dots,M} \omega_i(0). \qquad 0 < v_1^{\mathbf{max}} \le v_2^{\mathbf{max}} \le \dots \le v_M^{\mathbf{max}}.$$

$\psi : \mathbb{R}^+ \to \mathbb{R}^+$ smooth $\qquad \psi' \le 0$ s.t. $\psi(0) = 1$ and $\psi(r) = 0$ for $r \ge 1$.

[5] [Chiarello-Goatin, 2019, NHM]

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- Local multi-class model: Benzoni-Colombo, European J. Appl. Math., 2003

[5] [Chiarello-Goatin, 2019, NHM]

Existence of weak solutions locally in time[6]

Theorem

Let $\rho_i^0(x) \in (BV \cap \mathbf{L}^\infty)(\mathbb{R};\mathbb{R}^+)$, for $i = 1,\ldots,M$, and assumptions **(H)** *hold. Then the Cauchy problem*

$$\begin{cases} \partial_t \rho_i(t,x) + \partial_x \left(\rho_i(t,x) v_i((r * \omega_i)(t,x))\right) = 0, & i = 1,\ldots,M, \\ \rho_i(0,x) = \rho_i^0(x). \end{cases}$$

admits a weak solution on $[0,T[\,\times\mathbb{R}$, for some $T > 0$ sufficiently small.

Proof.

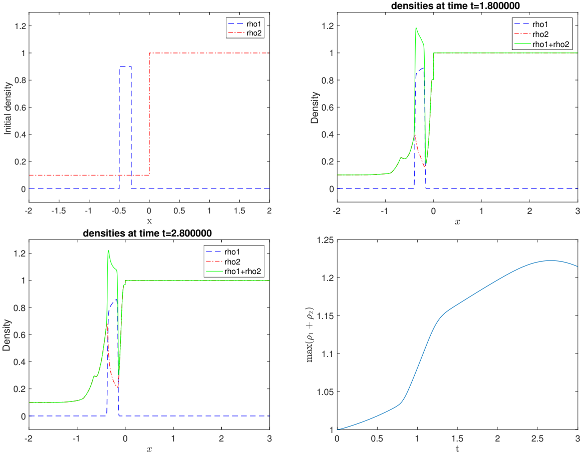
Helly's theorem + Lax-Wendroff type argument.

[6] [Chiarello-Goatin, 2019, NHM]

Unlike the classical multi-population model[7] , the simplex

$$\mathcal{S} := \left\{ \boldsymbol{\rho} \in \mathbb{R}^M : \sum_{i=1}^{M} \rho_i \leq 1, \ \rho_i \geq 0 \text{ for } i = 1, \ldots, M \right\}$$

is not an invariant domain for the non-local multi-class model.



[7] [Benzoni-Colombo, European J. Appl. Math, 2003]

Cars and trucks mixed traffic[8]

$$\begin{cases}\partial_t \rho_1(t,x) + \partial_x \left(\rho_1(t,x) v_1^{\max} \psi((r * \omega_1)(t,x))\right) = 0, \\ \partial_t \rho_2(t,x) + \partial_x \left(\rho_2(t,x) v_2^{\max} \psi((r * \omega_2)(t,x))\right) = 0,\end{cases}$$

with

$$\omega_1(x) = \frac{2}{\eta_1}\left(1 - \frac{x}{\eta_1}\right), \qquad \eta_1 = 0.3,$$

$$\omega_2(x) = \frac{2}{\eta_2}\left(1 - \frac{x}{\eta_2}\right), \qquad \eta_2 = 0.1,$$

$$\psi(\xi) = \max\{1 - \xi, 0\}, \qquad \xi \geq 0,$$

$$v_1^{max} = 0.8, \quad v_2^{max} = 1.3.$$

$$\begin{cases}\rho_1(0,x) = 0.5\chi_{[-1.1,-1.6]}, & \textit{trucks} \\ \rho_2(0,x) = 0.5\chi_{[-1.6,-1.9]}, & \textit{cars}\end{cases}$$

[8] [Chiarello-Goatin, 2019, NHM]

Cars and trucks mixed traffic

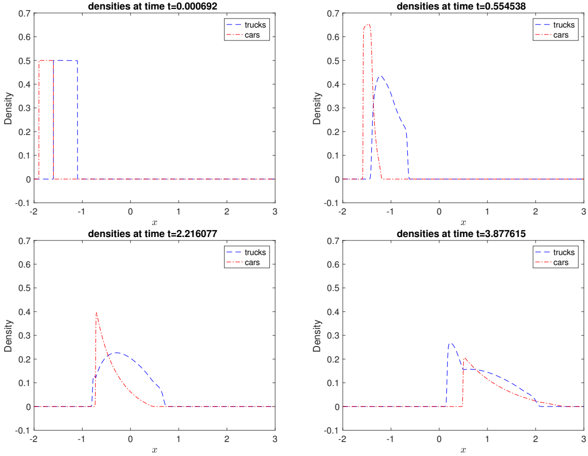


Figure: Density profiles of cars and trucks at increasing times corresponding to the non-local model.

Cars and trucks mixed traffic

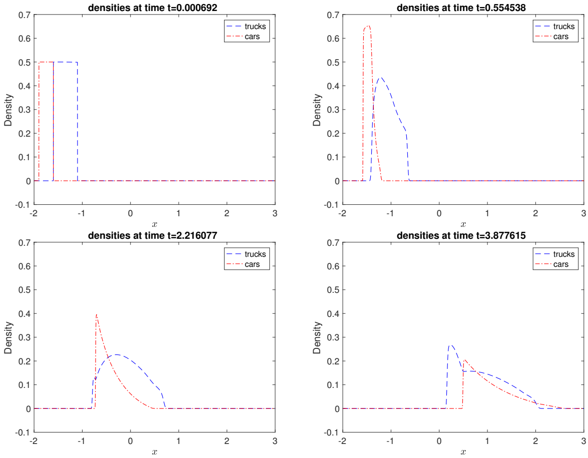


Figure: Density profiles of cars and trucks at increasing times corresponding to the non-local model.

Thanks for your attention!