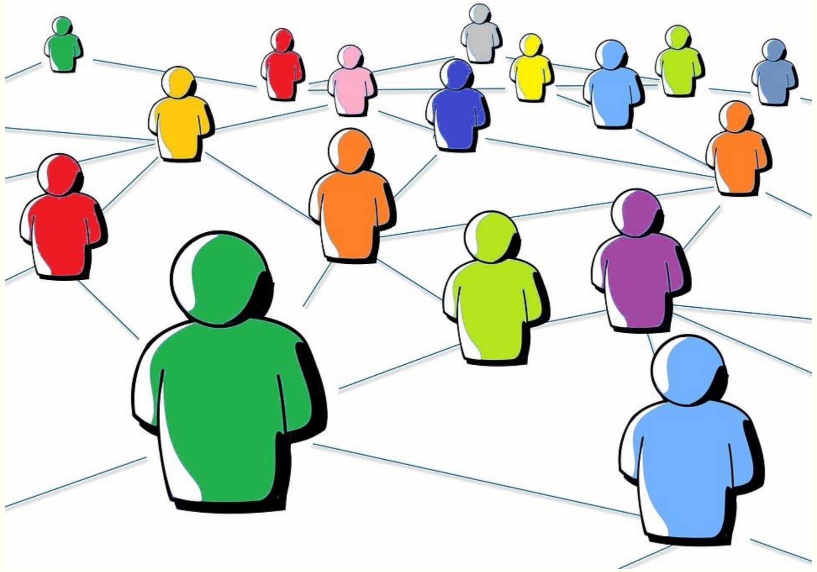
CONVERGENCE PROPERTIES OF SOCIAL HEGSELMANN-KRAUSE DYNAMICS



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For more details, see arXiv:1909.03485

Hegselmann-Krause Dynamics [Hegselmann, Krause 2002]

- n agents with initial opinions in $\mathbb{R}$
- State at time $k = x[k] = \left[x_1[k] \;\; x_2[k] \;\; \ldots \;\; x_n[k]\right]^T \in \mathbb{R}^n$ for $k = 0, 1, 2, \ldots$
- Confidence bound R:

$$N_i[k] = \left\{j \in [n]: \left|x_i[k] - x_j[k]\right| \leq R\right\}$$

Communication graph $G_c[k]$

- Update rule:

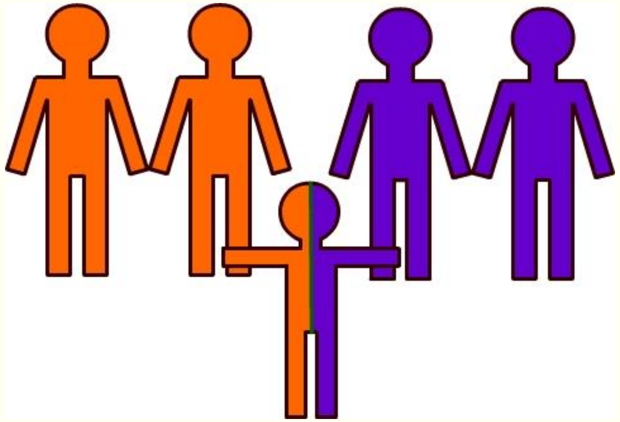
$$x_i[k+1] = \frac{\sum_{j \in N_i[k]} x_j[k]}{|N_i[k]|}$$

- State-dependent due to feedback between the state and the communication graph

A Shortcoming

If two agents have similar opinions, they necessarily influence each other.

What if two like-minded agents never meet?



Social Hegselmann-Krause Model [Fortunato 2005]

- Incorporate a physical connectivity graph G_{ph} that connects only those pairs of agents that can contact each other
- Properties of G_{ph}:
  - Static graph independent of agents' opinions (simplifying assumption)
  - Connected graph

Thus, two agents influence each other iff

1. their opinions are similar
2. they are adjacent in G_{ph}

$$\text{Influence graph} = G_{ph} \cap G_c[k]$$

How long does a social HK system take to reach the steady state?

(The original HK system takes $O(n^3)$ steps, [Mohajer, Touri 2013])

Definitions

- **Steady State:** if initial state = x_0, then $x_\infty(x_0) \coloneqq \lim_{k\to\infty} x[k]$ (always exists, [Lorenz 2005])
- **Termination Time**: time taken to reach the steady state

$$T(G_{ph}; x_0) \coloneqq \inf\{k \in \mathbb{N}: x[k] = x_\infty(x_0)\}$$

- **Maximum Termination Time:**

$$T^*(G_{ph}) \coloneqq \sup_{x_0 \in \mathbb{R}^n} T(G_{ph}; x_0)$$

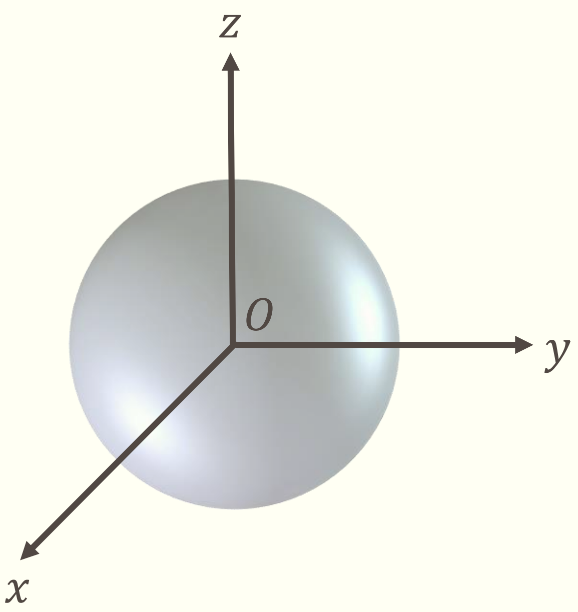
Analysis of Termination Time

Theorem 1: Suppose G_{ph} is *not* a complete graph. Then there exists $\rho > 0$ such that for all

$$x_0 \in B_0(\rho) := \{x \in \mathbb{R}^n : ||x||_2 \leq \rho\},$$

we have

$$T(G_{ph}; x_0) = \infty.$$



[Parasnis, Franceschetti, Touri. IEEE-CDC 2018.]

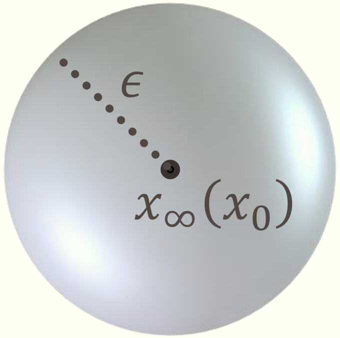
Consequently, $T^*(G_{ph}) = \infty$ if G_{ph} is not complete.

How fast does a social HK system approach the steady state?

Definitions

- **ϵ-Convergence time:** time taken to enter the ϵ-neighborhood of $x_\infty(x_0)$:

$$k_\epsilon\left(G_{ph}; x_0\right) \coloneqq \inf\left\{N \in \mathbb{N}: \left|\left|x[k] - x_\infty(x_0)\right|\right|_2 < \epsilon \ \ \forall\, k \geq N\right\}$$



- **Maximum ϵ-convergence time:**

$$k_\epsilon^*\left(G_{ph}\right) \coloneqq \sup_{x_0 \in \mathbb{R}^n} k_\epsilon\left(G_{ph}; x_0\right)$$

$$= \sup_{\left|\left|x_0\right|\right|_\infty \leq nR} k_\epsilon\left(G_{ph}; x_0\right)$$

Bounds on the Maximum Convergence Time

Theorem 2 (Lower Bound): Given an incomplete G_{ph} and an $\epsilon > 0$,

$$k_\epsilon^*(G_{ph}) > \frac{\log\left(\frac{R}{\epsilon\sqrt{2}}\right)}{\left|\log\left(1 - 2\phi(G_{ph})\right)\right|}.$$

[Parasnis, Franceschetti, Touri. IEEE-CDC 2018.]

Theorem 3 (Conditional Upper Bound): Let $x[0]$ be such that the influence graph, $G_{ph} \cap G_c[k]$ is connected and constant. Then the system achieves ϵ-convergence in $O\left(n^2 d(G_{ph}) \log n\right)$ steps.

Arbitrarily Slow Convergence to the Steady State

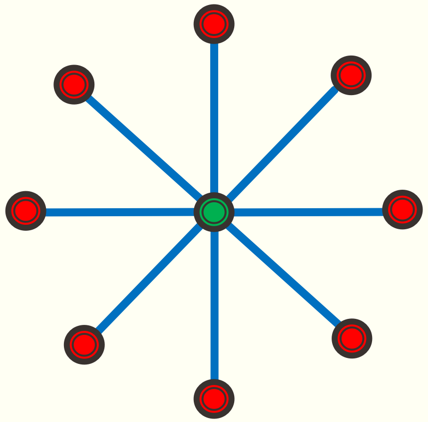
Fact: For every $n \geq 4$ and $\epsilon < R/2$, there exists a physical connectivity graph G_{ph} on n vertices for which $k_\epsilon^*(G_{ph}) = \infty$.

Theorem: Complete r-Partite Graphs Do Not Exhibit A.S.C.

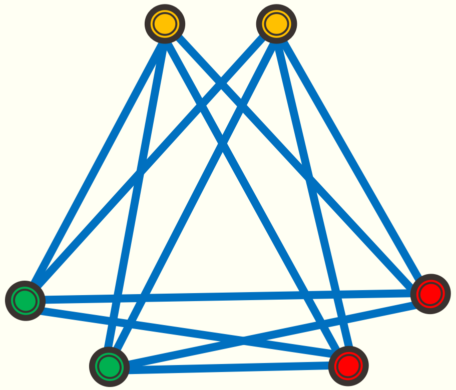
Theorem: Let $\epsilon > 0$ and $r \in \mathbb{N}$. Let G_{ph} be a complete r-partite graph. Then $k_\epsilon^*(G_{ph}) < \infty$.

[To appear in IEEE-CDC 2019.]

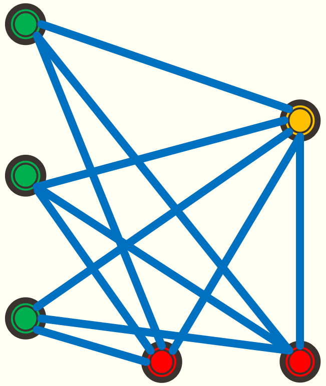
$\exists$ partition $\{V_1, \ldots, V_r\}$ of $V(G_{ph})$ such that $(i,j) \in E(G_{ph}) \Leftrightarrow i \in V_x, j \in V_y$ for some $x \neq y$



A Star Graph



A Turan Graph

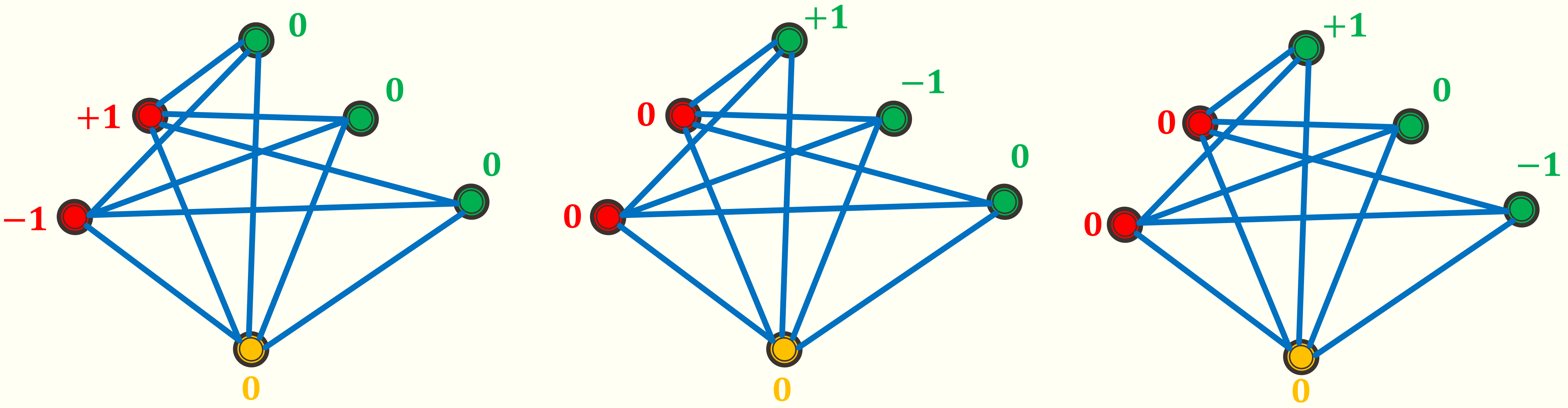


Parts of Different Sizes

By-product: A Graph-theoretic Lemma

A lemma characterizing the eigenvectors of the normalized adjacency matrices ($D^{-1}A_{adj}$) and hence those of the normalized Laplacian matrices ($D^{-\frac{1}{2}}A_{adj}D^{\frac{1}{2}}$) of complete r-partite graphs

Example: 3 eigenvectors of the **complete tripartite graph** with parts of sizes 1, 2 and 3



$\lambda_1 = \frac{1}{5}, \lambda_2 = \lambda_3 = \frac{1}{4}$

and 3 other eigenvectors...

Conclusions

	Original HK	Social HK
G_{ph}	Complete	Arbitrary
Maximum Termination Time	$O(n^3)$ [Mohajer, Touri 2013]	∞ for incomplete $\boldsymbol{G_{ph}}$
Maximum ϵ-Convergence Time	$O(n^3)$	• Possibly ∞ • finite for complete r-partite graphs

Thus, HK dynamics over complete graphs are an anomaly!

Future direction: Which other classes of graphs satisfy $k_\epsilon^*(G_{ph}) < \infty$?

THANK YOU!
QUESTIONS?

For more details, see
On the Convergence Properties of Social Hegselmann-Krause Dynamics **(arXiv:1909.03485)**