Controlling Spreading Processes in Networks
(or how to rapidly spread mutants throughout a geographic region)

Lorenzo Zino

DISMA Weekly Seminar
(Seminari Cadenzati - Progetto Eccellenza)

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Why to study spreading processes?

...because they are ubiquitous!
Why networks?

...powerful tool to represent interconnected systems!
Scientia potestas est
Knowledge is power [attributed to F.T. Bacon]


FDA approves Zika-fighting genetically modified mosquito

Genetically modified mosquitoes to help control dengue, malaria

U.S. One Step Closer to Releasing Engineered Mosquito to Fight Zika

Cases of Dengue Drop 91 Percent Due to Genetically Modified Mosquitoes

When Will The Keys Begin Using Genetically Modified Mosquitoes To Fight Zika?
Our goals

**Model** the phenomenon using evolutionary dynamics [Lieberman et al., Nature, 2005] with a new approach, which enables to perform **analysis** and include **control**

Understand how the **eradication time** and the **control cost** are influenced by the **topology** of the geographic region and by the **insertion policy** used

Design and study a control strategy to **speed up** the eradication of disease-spreading mosquitoes using a **feedback control policy**
Model
Geographic region

Weighted connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$:

- **Nodes** $\mathcal{V} = \{1, \ldots, n\}$ represent locations
- Undirected **links** and **weights** represent (heterogeneous) connections
State variables

- $X_i(t) \in \{0, 1\}$ state of node $i$ at time $t \geq 0$:
  $$X_i(t) = \begin{cases} 1 & \text{if } i \text{ occupied by mutant at time } t \\ 0 & \text{if } i \text{ occupied by native species at time } t \end{cases}$$

- Two observables:
  - $Z(t) = \sum_i X_i(t)$ number of locations occupied by mutants
  - $B(t) = \sum_i \sum_j X_i(t)(1 - X_j(t))W_{ij}$ boundary between the two species

```
Z(t) = 5  
B(t) = 28
```
Spreading mechanism

- **Poisson clocks** for activations of undirected links, rates from $W$

  \{i,j\} activates $\implies$ **conflict**: mutants win w.p. $\beta > 1/2$
  
  (evolutionary advantage)

- Different interpretation link-based **voter model** [Clifford and Sudbury, Biometrika, 1973] $\implies$ Application in opinion dynamics
External control

- A **target set** $M(t) \subseteq \mathcal{V}$ is selected
- Mutants **introduced** in $m \in M(t)$, rate $u_m(t) > 0$ [Harris et al., Nature Biotechnology, 2011]
- **Remark** Target set and rates can be constant or time-varying
- **Different interpretation** fictitious **stubborn** node $s$ with $X_s = 1$ linked to $m \in M$ [Acemoglu et al., Mathematics of Operations Research, 2013]
Markov jump process

- \( X(t) \) **Markov jump process** with \( X(0) = 0 \).

\[
\begin{align*}
\lambda_i^+(x) &= (1 - x_i) \left[ \beta \sum_j W_{ij} x_j + u_i(t) \right] \\
\lambda_i^-(x) &= x_i (1 - \beta) \sum_j W_{ij} (1 - x_j)
\end{align*}
\]

- \( X = 1 \) unique absorbing state; expected **eradication time** and **cost**:

\[
\tau = \mathbb{E} \left[ \inf t : X(t) = 1 \right] \quad J = \mathbb{E} \left[ \int_0^\infty \mathbb{1}^T u(t) \, dt \right]
\]
Markov jump process

- $X(t)$ Markov jump process with $X(0) = 0\mathbb{1}$.

\[
\begin{cases}
\lambda_i^+(x) = (1 - x_i) \left[ \beta \sum_j W_{ij} x_j + u_i(t) \right] \\
\lambda_i^-(x) = x_i (1 - \beta) \sum_j W_{ij} (1 - x_j)
\end{cases}
\]

- $X = \mathbb{1}$ unique absorbing state; expected eradication time and cost:

\[
\tau = \mathbb{E} \left[ \inf t : X(t) = \mathbb{1} \right] \quad J = \mathbb{E} \left[ \int_0^\infty \mathbb{1}^T u(t) dt \right]
\]

given a budget, how long does it take for eradication?
Analysis
Neighborhood monotone crusade

**Weighted boundary** of $\mathcal{W} \subseteq \mathcal{V}$: $B[\mathcal{W}] = \sum_{i \in \mathcal{W}} \sum_{j \notin \mathcal{W}} W_{ij}$

\[ B[S] = 19 \]
**Weighted boundary** of $W \subseteq V$: $B[W] = \sum_{i \in W} \sum_{j \notin W} W_{ij}$

**Neighborhood monotone crusade** from $W$: $S$ sequence of sets satisfying $S_0 = W$, and $S_i = S_{i-1} \cup \{v_i\}$, with $v_i \in N_{S_{i-1}} \setminus S_{i-1}$

$B[S_0] = 9$
Neighborhood monotone crusade

- **Weighted boundary** of $\mathcal{W} \subseteq \mathcal{V}$: $B[\mathcal{W}] = \sum_{i \in \mathcal{W}} \sum_{j \notin \mathcal{W}} W_{ij}$

- **Neighborhood monotone crusade** from $\mathcal{W}$: $S$ sequence of sets satisfying $S_0 = \mathcal{W}$, and $S_i = S_{i-1} \cup \{v_i\}$, with $v_i \in \mathcal{N}_{S_{i-1}} \setminus S_{i-1}$

\[ B[S_1] = 5 \]
**Weighted boundary** of $\mathcal{W} \subseteq \mathcal{V}$: $B[\mathcal{W}] = \sum_{i \in \mathcal{W}} \sum_{j \notin \mathcal{W}} W_{ij}$

**Neighborhood monotone crusade** from $\mathcal{W}$: $S$ sequence of sets satisfying $S_0 = \mathcal{W}$, and $S_i = S_{i-1} \cup \{v_i\}$, with $v_i \in N_{S_{i-1}} \setminus S_{i-1}$

$B[S_2] = 7$
Neighborhood monotone crusade

- **Weighted boundary** of $\mathcal{W} \subseteq \mathcal{V}$: $B[\mathcal{W}] = \sum_{i \in \mathcal{W}} \sum_{j \notin \mathcal{W}} W_{ij}$

- **Neighborhood monotone crusade** from $\mathcal{W}$: $S$ sequence of sets satisfying $S_0 = \mathcal{W}$, and $S_i = S_{i-1} \cup \{v_i\}$, with $v_i \in N_{S_{i-1}} \setminus S_{i-1}$

![Diagram]

$B[S_3] = 7$
Neighborhood monotone crusade

- **Weighted boundary** of $W \subseteq V$: $B[W] = \sum_{i \in W} \sum_{j \notin W} W_{ij}$

- **Neighborhood monotone crusade** from $W$: $S$ sequence of sets satisfying $S_0 = W$, and $S_i = S_{i-1} \cup \{v_i\}$, with $v_i \in N_{S_{i-1}} \setminus S_{i-1}$

$$B[S_4] = 1$$
Weighted boundary of $W \subseteq V$: $B[W] = \sum_{i \in W} \sum_{j \notin W} W_{ij}$

Neighborhood monotone crusade from $W$: $S$ sequence of sets satisfying $S_0 = W$, and $S_i = S_{i-1} \cup \{v_i\}$, with $v_i \in \mathcal{N}_{S_{i-1}} \setminus S_{i-1}$

$B[S_5] = 0$
Neighborhood monotone crusade

- **Weighted boundary** of $\mathcal{W} \subseteq \mathcal{V}$: $B[\mathcal{W}] = \sum_{i \in \mathcal{W}} \sum_{j \notin \mathcal{W}} W_{ij}$

- **Neighborhood monotone crusade** from $\mathcal{W}$: $S$ sequence of sets satisfying $S_0 = \mathcal{W}$, and $S_i = S_{i-1} \cup \{v_i\}$, with $v_i \in N_{S_{i-1}} \setminus S_{i-1}$

- $\Omega_{\mathcal{W}}$ all NMCs from $\mathcal{W}$. **Lower bound** on duration of NMC from $\mathcal{W}$

$$\tau_{\mathcal{W}} := \min_{S \in \Omega_{\mathcal{W}}} \sum_{i=0}^{n-|\mathcal{W}|-1} \frac{1}{B[S_i]}$$
Theorem I: Fundamental limit on $\tau$ and $J$ [working paper]

Let $\mathcal{U} = \bigcup_{t \geq 0} M(t)$ be set of controllable nodes, then $\tau$ and $J$ satisfy:

$$
\tau \geq \min_{\mu \in M_J} \sum_{\mathcal{W} \subseteq \mathcal{U}} \mu_\mathcal{W} \mathcal{T}_\mathcal{W},
$$

where $\mu$ is a probability measure on the subsets of $\mathcal{U}$ that belongs to

$$
M_J := \left\{ \mu : \sum_{\mathcal{W} \subseteq \mathcal{U}} \mu_\mathcal{W} |\mathcal{W}| \leq J \right\}
$$

Remark. The larger is $J$, the more measures are included in $M_J$, the smaller is the bound on $\tau$ trade-off between cost and eradication time.

⚠️ Not always easy to compute/estimate $\mathcal{T}_\mathcal{W}$ and deal with $\mu$. 
Sketch of the proof

- **Main idea**: relation between cost \( \int_0^\infty 1^T u(t) dt \) and set of controlled nodes \( \mathcal{W} \), in which mutants are introduced.

- Cost \( \implies \) **probability measure** \( \mu \) for set of controlled nodes \( \mathcal{W} \).

- **Stochastic domination** techniques: \( \mathcal{W} \) controlled nodes \( \implies \mathcal{W} \) initial condition of \( Y(t) \) with \( \beta = 1, u = 0 \): \( \tau_Y \leq \tau \).

- **NMCs** are admissible trajectories of \( Y(t) \): \( \tau_Y | Y(0) = \mathcal{W} \geq \mathcal{I}_\mathcal{W} \).

- Expected cost \( \implies \) expected value \( \mu \) \( \implies \text{T1} \).
Example: slow diffusion on rings

- Comparable examples: $W_{ij} = 1/d_{avg}$ ($d_{avg}$ average degree)
- We set $W_{ij} = 1/2$
- We use $S \in \Omega \Rightarrow B[S_i] \leq |\mathcal{W}|$
Example: slow diffusion on rings

- Comparable examples: $W_{ij} = 1/d_{avg}$ ($d_{avg}$ average degree)
- We set $W_{ij} = 1/2$
- We use $S \in \Omega_{\mathcal{W}} \implies B[S_i] \leq |\mathcal{W}|$

$d_{avg} \geq \frac{1}{j} n - 1$

$T1 \implies \text{slow diffusion} \ \forall \ \text{control policy } \tau \geq \frac{1}{j} n - 1$

![Graphs showing time versus n with beta values of 0.7 and 0.8]
Effective control: \( C(t) := \sum_i (1 - x_i(t)) u_i(t) \)

**Theorem II: Upper bound on \( \tau \) [IFAC, 2017]**

If during the evolution of the process the following is verified:

\[ B(t) + C(t) \geq f(Z(t)), \]

then it holds

\[ \tau \leq \frac{\beta}{(2\beta - 1)f(0)} + \frac{1}{2\beta - 1} \sum_{z=1}^{n-1} \frac{1}{f(z)} \]

**Remark.** The function \( f(z) \) depends on the **topology** of the network and on the **control policy** adopted.
Sketch of the proof

- **Stochastic process** $Z(t)$:

  \[
  C(t) \quad 1 \quad \ldots \quad z \quad \ldots \quad n
  \]

  \[
  \beta B(t) + C(t) \quad (1 - \beta)B(t)
  \]

- **Stochastic domination**: $Z(t) \succeq \tilde{Z}(t)$

  \[
  C(t) \quad 1 \quad \ldots \quad \tilde{z} \quad \ldots \quad n
  \]

  \[
  \beta B(t) \quad (1 - \beta)B(t)
  \]

- **Discrete-time chain** $\tilde{Z}(k)$ (time step at each jump)

  \[
  1 \quad 1 \quad \ldots \quad \tilde{z} \quad \ldots \quad n
  \]

  \[
  1 - \beta \quad \beta
  \]
Sketch of the proof (cont’d)

- Process $\tilde{Z}(k)$ is Markov $\implies$ bound $\#$ of entrances

\[ \mathbb{E}[N_0] \leq \frac{\beta}{2\beta - 1} \quad \text{and} \quad \mathbb{E}[N_i] \leq \frac{1}{2\beta - 1} \]

- Time spent in state $Z(t) = z$ is **Poisson** random variable with rate

\[ \mu(t) = \lambda^+(t) - \lambda^-(t) = B(t) + C(t) \geq f(z) \]

- Expected **time spent** in each entrance in $z$ is $\mathbb{E}[T_z] \leq 1/f(z)$

- Bound on $\#$ of entrances in $z$ + bound on time spent in $z$ $\implies$ T2

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L. Zino, G. Como, and F. Fagnani, *Fast Diffusion of a Mutant in Controlled Evolutionary Dynamics*, IFAC-PapersOnLine, **50**(1), 11908-11913, 2017
Control
**Constant control policies**

- Target set $M(t) = M$ **time-invariant**, $u_i(t) = u_i$
- To avoid waste, mutants **not** introduced in nodes already occupied

$$ u_i(t) = \begin{cases} u_i & \text{if } X_i(t) = 0 \\ 0 & \text{if } X_i(t) = 1 \end{cases} $$

**Expected cost** $J$ proportional to $|M| \implies$ study eradication time $\tau$
The **bottleneck** of $G$ controlled in $M$ is $\bar{\phi}_M = \min_{M \subseteq \mathcal{W} \subseteq V} B[\mathcal{W}]$. 

![Diagram showing a graph with labeled edges and a shaded area indicating a constant control policy.](image-url)
Constant control policy: performance (LB)

The **bottleneck** of $G$ controlled in $M$ is $\bar{\phi}_M = \min_{M \subseteq \mathcal{W} \subseteq \mathcal{V}} \mathcal{B}[\mathcal{W}]$.

**Corollary:** Lower bound on $\tau$ under constant control (LB)

$$\tau \geq \frac{\bar{\phi}_M - 1}{\bar{\phi}_M}$$
Constant control policy: performance (UB)

Minimum conductance profile: \( \phi(z) = \min_{W \subset \mathcal{V}, |W| = z} \mathcal{B}[W] \)

\[
\begin{align*}
\phi(3) &= 11
\end{align*}
\]
**Constant control policy: performance (UB)**

**Minimum conductance profile:** $\phi(z) = \min_{\mathcal{W} \subseteq \mathcal{V}, |\mathcal{W}| = z} B[\mathcal{W}]$

![Graph with nodes and edges labeled with numbers and weights]

**Corollary:** Upper bound on $\tau$ under constant control (UB)

$$\tau \leq \frac{1}{2\beta - 1} \sum_{z=1}^{n-1} \frac{1}{\phi(z)} + \frac{\beta}{(2\beta - 1)\mathbf{1}^T \mathbf{u}}$$
Example: fast diffusion on expander graphs

- $\exists \gamma > 0 : \phi(z) \geq \gamma \min\{z, n - z\}$ (complete, ER, small-world,...)
- We set $M = \{1\}$ and $W_{ij} = 1/d_{avg}$

Complete

Erdős-Rényi
Example: fast diffusion on expander graphs

- $\exists \gamma > 0 : \phi(z) \geq \gamma \min\{z, n - z\}$ (complete, ER, small-world, ...)
- We set $M = \{1\}$ and $W_{ij} = 1/d_{avg}$
- $UB + T1 \iff$ fast diffusion: $\tau = \Theta(\ln n)$
Example: slow diffusion on stochastic block models

- ER communities linked by few edges (extreme case: barbell)
- We set $M = \{1\}$ and $W_{ij} = 1/d_{avg}$ $\forall (i, j) \in \mathcal{E}$
Example: slow diffusion on stochastic block models

- ER communities linked by few edges (extreme case: barbell)
- We set $M = \{1\}$ and $W_{ij} = 1/d_{avg}$ $\forall (i, j) \in \mathcal{E}$
- UB + LB $\implies$ **Slow diffusion**: $\tau = \Theta(n)$

\begin{itemize}
  \item $\beta = 0.7$
  \item $\beta = 0.8$
\end{itemize}
To sum up...

😊 If the topology ensures fast diffusion:

we reached our goal!

😊 If it does not:

can we speed up the spread?

⇒ Improve the control policy
Feedback control policy

Assumptions:
- Target set $M(t) = \text{target node } m(t)$, moved during the dynamics
- Control rate feedback function of the two observables $Z(t), B(t)$

Avoid waste: $m(t)$ moved so that it is always a node with state 0
$\implies$ no optimization on $m(t)$ needed (computationally good!)

Contrast slowdowns: velocity of the process proportional to $B(t)$
$\implies u(t)$ should compensate when $B(t)$ is small

$$u(t) = u(Z(t), B(t)) = \begin{cases} C - B(t) & \text{if } Z(t) \neq n, B(t) < C \\ 0 & \text{else} \end{cases}$$
Feedback control policy: performance

Bounds on expected eradication time and expected control cost:

\[ \tau = \mathbb{E} \left[ \inf t : X(t) = 1 \right] \quad J = \mathbb{E} \left[ \int_0^\infty 1^T u(t) dt \right] \]

Corollary: upper bounds on \( \tau \) and \( J \) under feedback control (UB-F)

\[ \tau \leq \frac{\beta}{(2\beta - 1)C} + \frac{1}{2\beta - 1} \sum_{z=1}^{n-1} \frac{1}{\max\{\phi(z), C\}} \]

\[ J \leq \frac{\beta}{(2\beta - 1)} + \frac{1}{2\beta - 1} \left| \{z : \phi(z) < C\} \right| \]
Application of feedback control policy to SBM

- $C < \frac{1}{2} \left(1 - \frac{2}{n}\right) \implies$ control activates only at **bottleneck**

- $T_{1+UB-F} \implies$ **fast diffusion:** $\tau \in \Theta(\ln n)$, $J \in \Theta(1)$

\[
\tau \leq \frac{2}{2\beta - 1} \ln n + \frac{2 + \beta}{C(2\beta - 1)} \\
J \leq \frac{2 + \beta}{2\beta - 1}
\]

**Constant control policy** vs. **Feedback control policy**

![Graphs showing time vs. n for constant and feedback control policies](image-url)
Case Study: Zika in Rwanda

Zika Alert in Rwanda since 2016 [CDC Centers for Disease Control and Prevention, available online]
Case Study: Zika in Rwanda

Zika Alert in Rwanda since 2016 [CDC Centers for Disease Control and Prevention, available online]
Case Study: Zika in Rwanda

- Location connected within a certain distance. Threshold set to 11.7 km: max distance traveled by mosquitoes to lay eggs [Bogojevic et al., J.Amer.Mosq.Cont.Ass., 2007]

- Activation rate $w = 1/10$. 10 days life-cycle of Aedes aegypti [CDC Centers for Disease Control and Prevention, available online]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Meaning</th>
<th>Value</th>
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<tr>
<td>$n$</td>
<td>Number of locations</td>
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<tr>
<td>$W_{ij}$</td>
<td>Activation rate</td>
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<td>$\beta$</td>
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<td>$u$</td>
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<td>$C$</td>
<td>Control parameter (feedback)</td>
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<tr>
<td>$t$</td>
<td>Time unit</td>
<td>day</td>
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Case Study: Zika in Rwanda

Constant vs Feedback
Case Study: Zika in Rwanda

![Graphs showing time and cost comparisons between constant and feedback scenarios.](image)

😊 Same cost, **performance improved**: $\tau \downarrow -56\%, p\text{-value} \ll 0.001$

L. Zino, G. Como, and F. Fagnani, *Controlling Evolutionary Dynamics in Networks: A Case Study*, accepted at IFAC-NecSys, Aug 2018
Conclusion and direction for future research

- Evolutionary dynamics with heterogeneity and exogenous control
- Model analyzable $\implies$ fundamental limit, performance estimation
- Constant control: in some topologies $\implies$ slow spread
- We designed a feedback control policy to speed up the process
- Validation on case study: Zika in Rwanda

$\implies$ Analyze Robustness (e.g., errors in measurement of $B(t)$)
$\implies$ Refine the analytical results to improve performance estimation
$\implies$ Plan and study feasibility in real-world applications
Thank you for your attention!

This work is a joint research project with...

Giacomo Como  
giacomo.como@control.lth.se

Fabio Fagnani  
fabio.fagnani@polito.it