Optimal intervention in transportation networks

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Joint work with Giacomo Como, Asu Ozdaglar, Francesca Parise

Workshop on algorithmic game theory, mechanism design, and learning.

November 10, 2022

Politecnico di Torino

Motivation

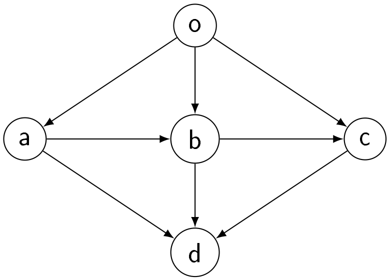
- Congestion of transportation networks leads to pollution and massive waste of time and money
- Need for analysis and design of transportation networks
- Intervention must take into account strategic user decisions

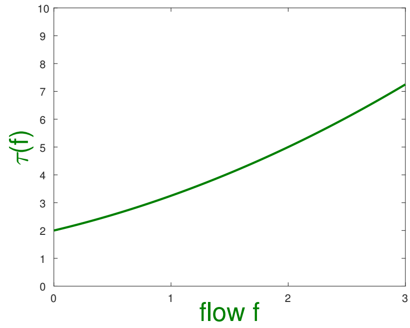
Part I: Model and problem formulation

Routing games: model

Network

- Transportation network as directed multigraph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$
- Non-decreasing delay function $\tau_e(f_e)$ for every link e
- Single origin-destination pair (o, d)
- Throughput m





Wardrop equilibria

- User behaviour $\rightarrow$ population game theory (user set is a continuum)
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- $f^{(0)}$ Wardrop equilibrium iff

$$f^{(0)} = \underset{\text{flow } f}{\arg\min} \sum_{e \in \mathcal{E}} \int_0^{f_e} \tau_e(s) ds$$

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- $f^{(0)}$ Wardrop equilibrium iff

$$f^{(0)} = \arg\min_{\text{flow } f} \sum_{e \in \mathcal{E}} \int_0^{f_e} \tau_e(s) ds$$

- Existence of at least an equilibrium
- If τ strictly increasing $\implies$ uniqueness of equilibrium

Intervention in transportation: two classes of strategies

First strategy

Influence indirectly user behaviour to align Wardrop equilibria with system-optimum flows:

- Tolls [Sandholm '02, Fleischer '04, Cole '06]
- Information design [Das '17, Wu '19, Zhu '22]

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Second strategy

Intervention on the infrastructure:

- Network design problem (NDP) [LeBlanc '75, Yang '98, Farahani '13]

NDP: problem formulation

- Write delay functions as $\tau_e(f_e) = \tau_e(0) + a_e(f_e)$ for every link e

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- Let system planner design intervention vector $u \in \mathbb{R}_+^{\mathcal{E}}$ with cost

$$\Psi(u) = \sum_{e \in \mathcal{E}} \psi_e(u_e)$$

s.t. delay functions become

$$\tau_e^{(u_e)}(f_e) = \tau_e(0) + \frac{a_e(f_e)}{1 + u_e}$$

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- Wardrop equilibrium after intervention

$$f^{(u)} = \underset{\text{flow } f}{\arg\min} \quad \sum_{e\in\mathcal{E}} \int_0^{f_e} \tau_e^{(u_e)}(s)ds$$

Network interventions

- Intervention vector $u \in \mathbb{R}_+^{\mathcal{E}}$ with cost $\Psi(u) = \sum_{e \in \mathcal{E}} \psi_e(u_e)$
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- Optimal intervention problem

$$\min_{u \in \mathbb{R}_+^{\mathcal{E}}} T(u) + \Psi(u)$$

NDP vs toll design

- In NDP, TTT evaluated w.r.t. new delay functions and new Wardrop eq., i.e.,

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Assumptions

- Affine routing game (i.e., $\tau_e(f_e) = a_e f_e + b_e$) (will be relaxed in numerical analysis)
- Assume intervention on single link (i.e., $u = \kappa\delta^{(e)}$), so interventions are pairs (e, κ).

Goal

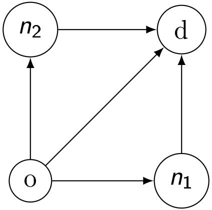
Find optimal intervention $(e, \kappa)^*$

Part II: Electrical interpretation

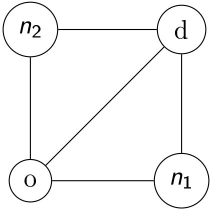
A related resistor network

- Transportation network with affine delays $\tau_e(f_e) = a_e f_e + b_e$
- Undirected resistor network: same node set $\mathcal{N}$
- Link $\{i, j\}$ in resistor network if there exists either $e = (i, j)$ or $e = (j, i)$ with $f_e^{(0)} > 0$
- Conductance of link $\{i, j\}$ is $W_{ij} = \frac{1}{a_e}$

Transportation network



Resistor network



Electrical interpretation

- y denotes electrical current on links when net current m is injected from o to d
- For a link $e = (i, j)$, r_e denotes effective resistance between i and j, i.e.,

$$r_e = x_i - x_j$$

where x voltage vector when unitary current from i to j is injected, i.e.,

$$\sum_k W_{hk}(x_h - x_k) = \delta^{(i)} - \delta^{(j)} \qquad \forall h \in \mathcal{N}$$

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Theorem 1 [Cianfanelli,Como,Ozdaglar,Parise,'22]

- Assumption: support of Wardrop eq. not modified with intervention
- Result: intervention (e, κ) yields TTT variation

$$\Delta T(e, \kappa) = a_e f_e^{(0)} \frac{y_e}{\frac{1}{\kappa} + \frac{r_e}{a_e}}.$$

Some observations

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Assumption holds if:

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Complexity

- $f^{(0)}$ observable
- vector y computed by solving a sparse linear system
- r_e to be computed for every link ($|\mathcal{E}|$ linear systems to be solved)

Part III: Effective resistance approximation

Local approximation of effective resistance

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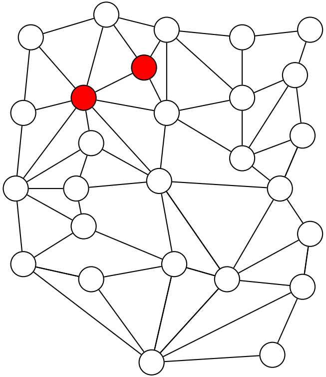
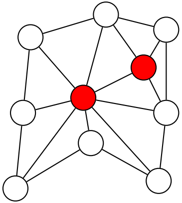
Idea: effective resistance can be locally approximated

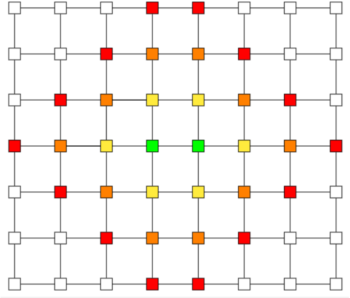


r=0.4606

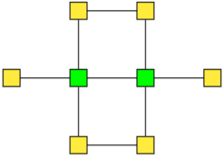


r=0.4871

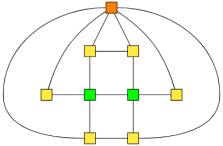
Cutting and shorting



Upper bound: cutting (distance 1)



Lower bound: shorting (distance 1)

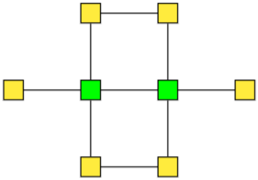


Upper and lower bound

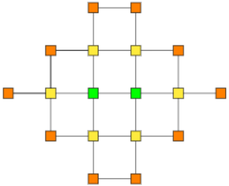
- r_e effective resistance
- $r_e^{U_d}$ computed on network cut at distance d
- $r_e^{L_d}$ computed on network shorted at distance d

Proposition [Cianfanelli, Como, Ozdaglar, Parise, '22]

- $r_e^{U_d} \geq r_e \geq r_e^{L_d} \quad \forall d \geq 1$
- if $d_1 < d_2 \implies \begin{cases} r_e^{U_{d_1}} \geq r_e^{U_{d_2}} \\ r_e^{L_{d_1}} \leq r_e^{L_{d_2}} \end{cases}$



$d = 1$



$d = 2$

Effective resistance bounds

- Can use Theorem 1 with $(r_e^{U_d} + r_e^{L_d})/2$ instead of r_e

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- Define random walk with rates W
- Definition: random walk is recurrent iff it visits its starting node infinite times with probability one (for infinite networks)

Proposition [Cianfanelli,Como,Parise,Ozdaglar,'22]

If random walk is recurrent, then for every link e

$$\lim_{d \to +\infty} r_e^{U_d} - r_e^{L_d} = 0$$

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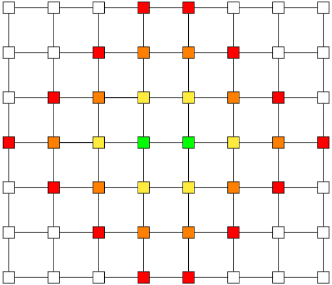
- 2d-grids and most of planar networks are recurrent
- Also for non-recurrent networks bounds often converge (for 3d-grids, $r_e^{U_d} - r_e^{L_d} = O(d^{-5/2})$)

Part IV: Numerical examples

Bound gap with finite distance

- Consider infinite square grid

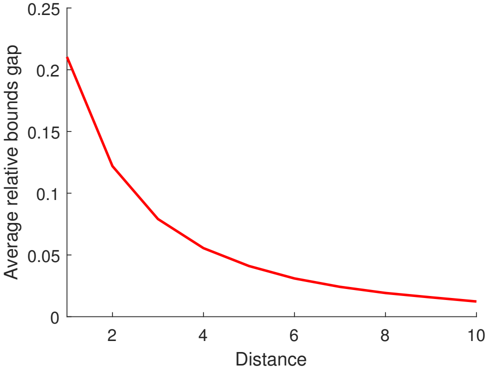


	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
$(r_e^{U_d} - r_e)/r_e$	$1/5$	0.0804	0.0426	0.0262	0.0178
$(r_e - r_e^{L_d})/r_e$	$1/5$	0.0804	0.0426	0.0262	0.0178

$$r_e^{U_d} - r_e = r_e - r_e^{L_d} = O(d^{-2})$$

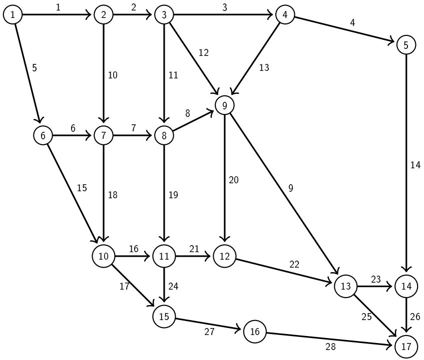
Case-study: Oldenburg transportation network

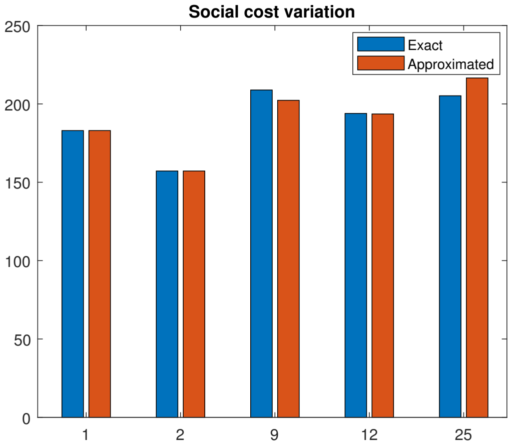
- $|\mathcal{N}| = 6105, |\mathcal{E}| = 7035$, diameter $= 104$
- Bounds computed at distance d:
  - $d = 4$, $|\mathcal{N}| \simeq 15$, relative bound gap $\simeq 0.06$
  - $d = 7$, $|\mathcal{N}| \simeq 45$, relative bound gap $\simeq 0.03$



Relaxing assumptions

- What if Wardrop support changes with intervention?

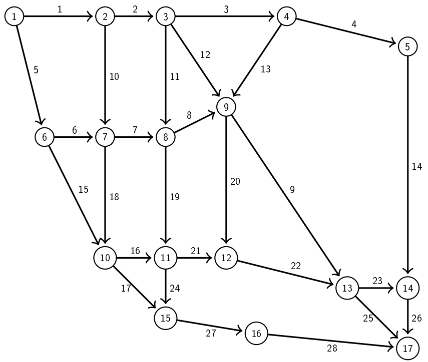


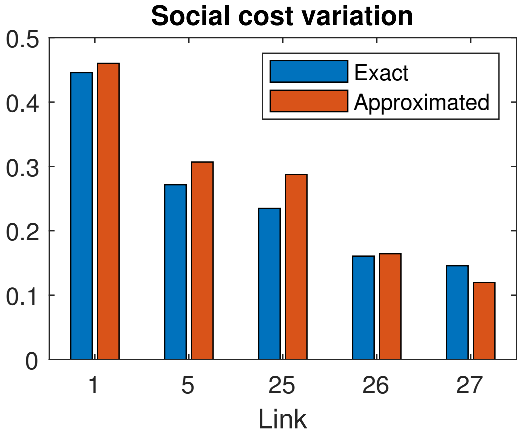


- Cost variation computed by convex optimization ($\kappa = 3$)
- Cost variation computed by our theorem, ignoring that Wardrop support varies

Relaxing assumptions

- Non-affine delay functions $\tau_e(f_e) = a_e f_e^4 + b_e$
- Construct resistor network with resistances $\frac{\tau_e(f_e^{(0)})}{f_e^{(0)}}$





- The method is validated for other functional forms of delay

Conclusions

Summary

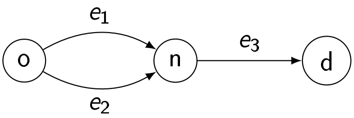
- Rephrased a NDP in terms of resistor networks
- Provided method to approximate the effective resistance between neighbors
- Constructed efficient algorithm to solve the NDP

Future research

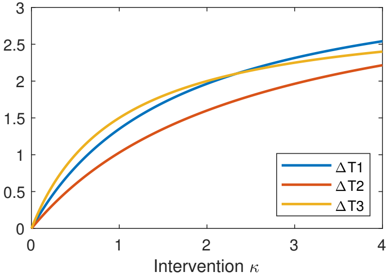
- Improve characterization of bound performance in order to optimize selection of distance d
- Relax assumptions:
  - Multiple origins and destinations
  - Intervention on multiple links
- Study different types of intervention, e.g., adding new links

L. Cianfanelli, G. Como, A. Ozdaglar, F. Parise, *Optimal intervention in transportation networks*, conditionally accepted to TAC (preliminary version: *arXiv preprint arXiv:2102.08441* (2021))

An example



$$\begin{cases} \tau_1(f_1) = 3f_1 \\ \tau_2(f_2) = 2f_2 \\ \tau_3(f_3) = f_3 \\ m = 3 \end{cases}$$



- No decoupling: the optimal link depends on κ!

Algorithm

- Construct the resistor network
- Compute y by solving sparse linear system (quasi-linear in $|\mathcal{N}|$)
- For every $e \in \mathcal{E}$
  - Compute $r_e^{U_d}$ and $r_e^{L_d}$
  - Select κ_e^* s.t.

$$\kappa_e^* \in \arg\max_{\kappa_e \geq 0} \quad a_e f_e^{(0)} \frac{y_e}{\frac{1}{\kappa_e} + \frac{r_e^{U_d} + r_e^{L_d}}{2a_e}} - \alpha \psi_e(\kappa_e).$$

- Select optimal link e^* s.t.

$$e^* \in \arg\max_{e \in \mathcal{E}} \quad a_e f_e^{(0)} \frac{y_e}{\frac{1}{\kappa_e^*} + \frac{r_e^{U_d} + r_e^{L_d}}{2a_e}} - \alpha \psi_e(\kappa_e^*).$$

- Optimal intervention is (e^*, κ_e^*)