DISMA Dipartimento di Scienze Matematiche G. L. Lagrange
ECCELLENZA 2018 · 2022

Target interventions in opinion dynamics

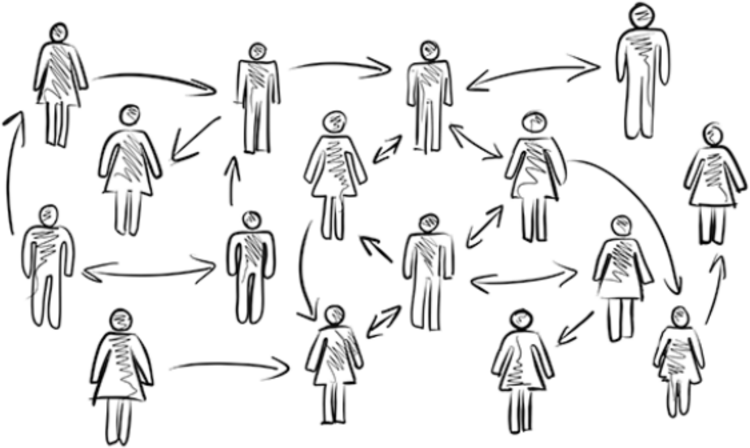
Luca Damonte

Algorithmic game theory, mechanism design, and learning

8-11 November 2022

POLITECNICO DI TORINO

Opinion formation process



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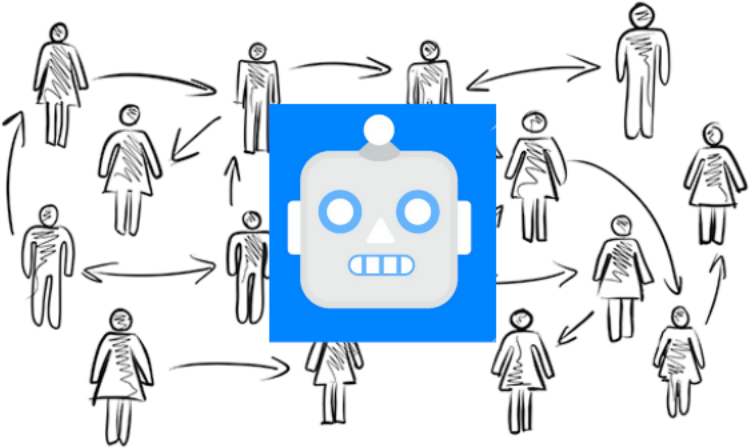

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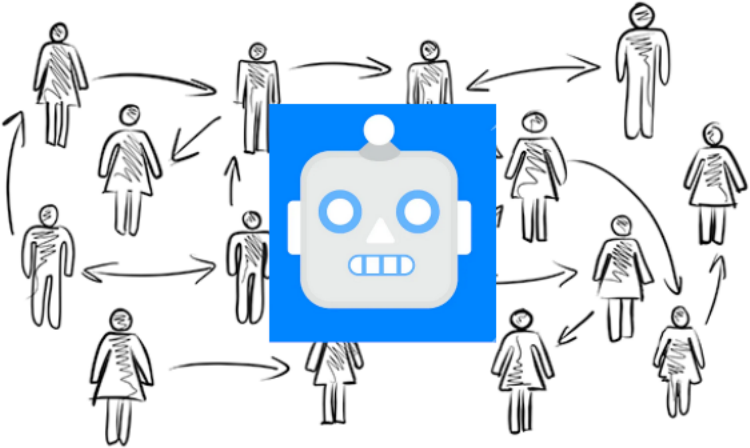

Characterize the **optimal intervention** for the defender.

The model

Linear time-invariant dynamics:

$$x(t+1) = Ax(t) + Bu\,, \qquad t \in \mathbb{N}\,,$$

- A in $\mathbb{R}_+^{n\times n}$ and B in $\mathbb{R}_+^{n\times m}$ are **nonnegative** matrices;
- u in $\mathbb{R}^m$ is a constant input vector;
- $x(t)$ in $\mathbb{R}^n$ represents the system state vector at time $t \geq 0$.

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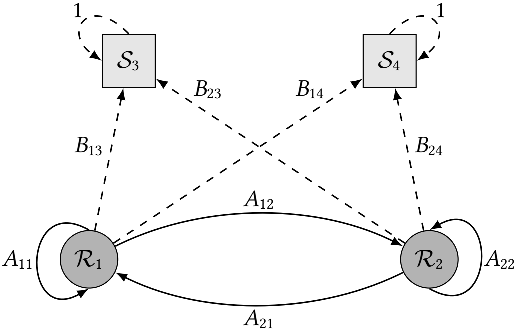
- $\mathcal{R} = \{1, 2, \ldots, n\}$ (regular) agents
- $\mathcal{S} = \{n+1, n+2, \ldots, n+m\}$ exogenous sources (**stubborn agents**)

$$G = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$$

directed graph $\mathcal{G} = (\mathcal{R} \cup \mathcal{S}, \mathcal{E})$
$\mathcal{E} = \{(i,j) | G_{ij} > 0\}$



Friedkin-Johnsen model

$$x(t+1) = [\lambda] W x(t) + (I - [\lambda]) x(0) \, .$$

$[\lambda] = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $A = [\lambda] W$, $B = (I - [\lambda])$.

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Assumption 1: A has spectral radius $\rho(A) < 1$.

$$\lim_{t\to\infty} x(t) = Mu\,,$$
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Assumption 2: the graph $\mathcal{G}$ is weakly connected.

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Defense-attack: u has the form

$$u = [\nu]^{-1/2}\omega,$$

- $\omega \in \mathbb{R}^m$ exogenous input (e.g. bot influence)
- $\nu \in \mathbb{R}^m_{++}$ defense intervention (e.g. mitigating factor)

Adversarial optimization problem

$$\Phi(\nu,\omega) = \|x\|^2 = \omega^\top [\nu]^{-1/2} H [\nu]^{-1/2} \omega\,, \quad H = M^\top M\,,$$
$$\min_{\nu \in \mathcal{Q}_c} \max_{\|\omega\|_2 \le 1} \Phi(\nu,\omega)\,. \tag{1}$$

Let $d \in \mathbb{R}^m_{++}$, $c \ge \mathbb{1}^\top d$. The set of **admissible interventions** is

$$\mathcal{Q}_c = \{\nu \in \mathbb{R}^m \mid \nu_i \ge d_i,\ \mathbb{1}^\top \nu \le c\}.$$

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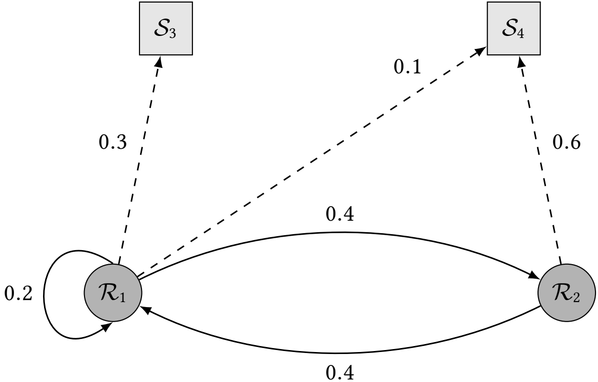
Adversarial Perturbations of Opinion Dynamics in Networks [1]

Set $\nu_i u_i^2 = \omega_i^2$, $c = n$, and assume $\nu > 0$, then min-max problem results

$$\min_{\substack{\nu > 0 \\ \sum_i \nu_i = n}} \max_{\substack{u \in \mathbb{R}^n \\ \|[\nu]^{1/2} u\|^2 = 1}} u^\top H u\,.$$

[1] Gaitonde, Jason, Jon Kleinberg, and Eva Tardos. "Adversarial perturbations of opinion dynamics in networks." Proceedings of the 21st ACM Conference on Economics and Computation. 2020.

Example



$$A = \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0.3 & 0.1 \\ 0 & 0.6 \end{pmatrix}.$$

Worst perturbation

Define the function $\phi : \mathcal{Q}_c \to \mathbb{R}$ as

$$\phi(\nu) = \max_{\|\omega\|_2=1} \omega^\top [\nu]^{-1/2} H [\nu]^{-1/2} \omega, \quad \nu \in \mathcal{Q}_c \,.$$

Lemma

Let d in $\mathbb{R}^m_{++}$ and $c \geq \mathbb{1}^\top d$. Then, for every ν in $\mathcal{Q}_c$, we have

(i) $\phi(\nu) = \rho\left([\nu]^{-1/2} H [\nu]^{-1/2}\right) = \rho\left(M [\nu]^{-1} M^\top\right)$.

Moreover, if H is irreducible, then:

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Moreover, if H is irreducible, then:

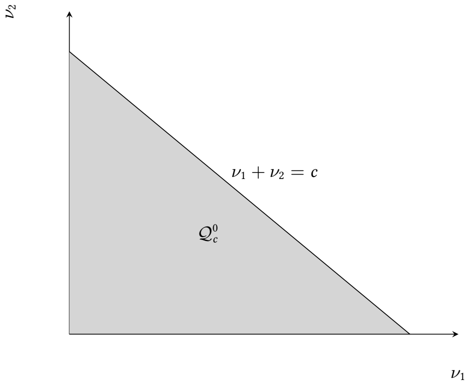
(ii) $\phi(\nu)$ *is strictly convex in* ν.

$$\min_{\nu \in \mathcal{Q}_c} \max_{\|\omega\|_2 \leq 1} \Phi(\nu, \omega) = \min_{\nu \in \mathcal{Q}_c} \rho\left(M [\nu]^{-1} M^\top\right) . \tag{2}$$

$$\nu^*(c) = \operatorname*{argmin}_{\nu \in \mathcal{Q}_c} \rho\left(M [\nu]^{-1} M^\top\right) .$$

Unbounded problem

$$\min_{\nu \in \mathcal{Q}_c^0} \phi(\nu)\,, \quad \mathcal{Q}_c^0 = \{\nu \in \mathbb{R}^m \mid \nu_i > 0,\ \mathbb{1}^\top \nu \leq c\}\,.$$



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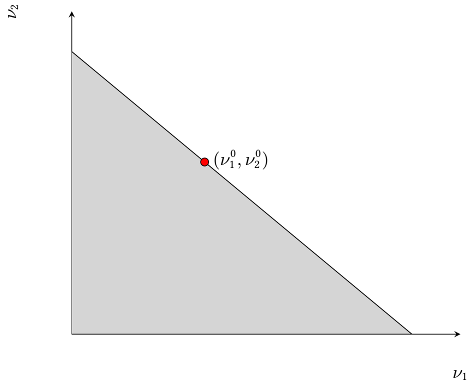


define the **centrality vector** $\pi = \frac{1}{\sum_{i,j} H_{ij}} H\mathbb{1}$

Proposition

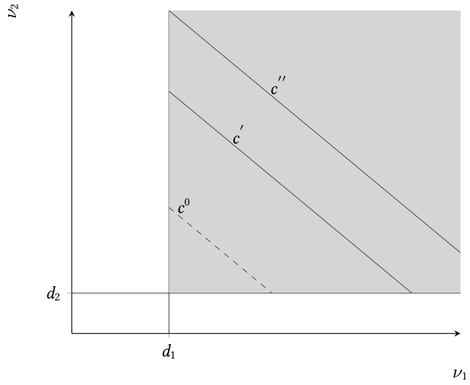
For every $c > 0$, we have that

$$\nu^0(c) = c\pi, \quad \min_{\nu \in \mathcal{Q}_c^0} \phi(\nu) = \phi(c\pi) = \frac{\mathbb{1}^\top H \mathbb{1}}{c}.$$

- $\nu_i^0(c) \propto \pi_i$, example $[0.328, 0.672]$;
- $\nu_i^0(c)$ increasing in c.

High budget regime

$$\nu^0(c) \in \mathcal{Q}_c \iff c \geq c^0 = \max_{i=1}^{m} \frac{d_i}{\pi_i}.$$



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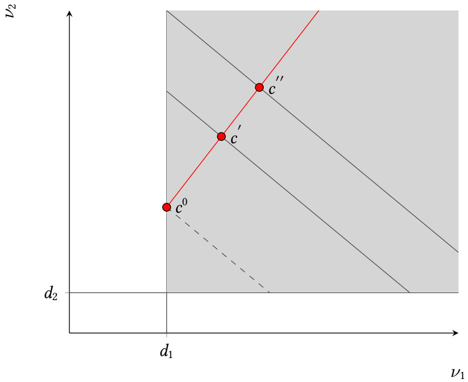


considering that $\mathcal{Q}_c \subseteq \mathcal{Q}_c^0$:

$$c \geq c^0 \implies \nu^*(c) = \nu^0(c)$$

Theorem

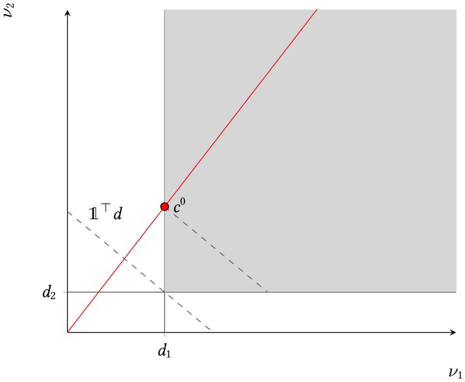
For every $d \in \mathbb{R}^m_{++}$ and $c \geq c^0$ we have

$$\nu^*(c) = c\pi, \quad \phi(\nu^*(c)) = \frac{\mathbb{1}^\top H \mathbb{1}}{c}. \qquad (3)$$

Saturated problem

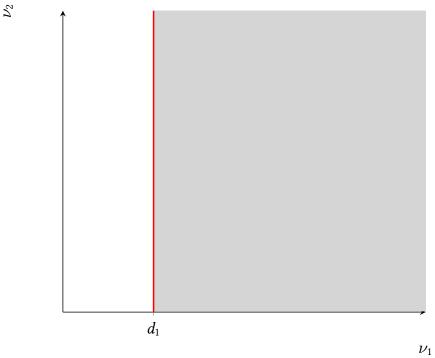
$$\mathbb{1}^\top d \leq c < c^0 \implies \nu^0(c) \notin \mathcal{Q}_c \implies \nu_i^*(c) = d_i \text{ for some } i \in \mathcal{S}\,.$$

$\nu^*(c)$ exhibits one or more components **saturated to their lower bound level**



Saturated problem

$$\phi(\nu^{\mathcal{U}}(c)) = \min_{\nu \in \mathcal{Q}_c^{\mathcal{U}}} \phi(\nu)\,, \qquad \begin{aligned} &\mathcal{W} = \{i \in \mathcal{S} \mid \nu_i = d_i\},\ \mathcal{S} = \mathcal{U} \cup \mathcal{W} \\ &\mathcal{Q}_c^{\mathcal{U}} = \{\nu \in \mathbb{R}^m_{++} \mid \nu_i = d_i\ \forall i \in \mathcal{W},\ \mathbb{1}^\top \nu \leq c\} \end{aligned}\,.$$



$$\begin{aligned} &\mathcal{W} = \{1\} \Rightarrow \nu_1 = d_1 \\ &\mathcal{U} = \{2\} \Rightarrow \nu_2 = c - d_1\,. \end{aligned}$$

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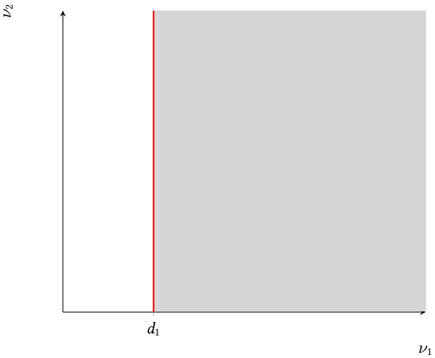

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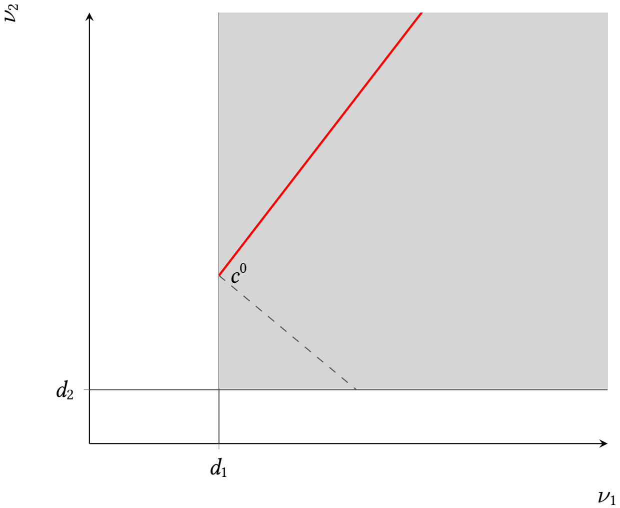
Proposition

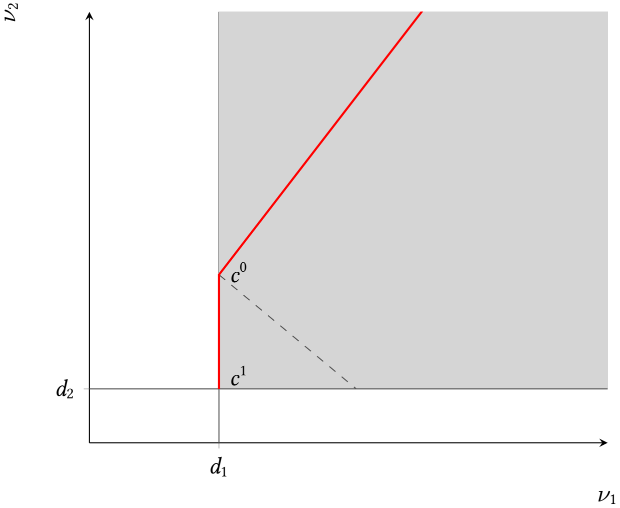
Fix $d \in \mathbb{R}^{\mathcal{W}}_{++}$. Then, for every $c \ge \mathbb{1}^\top$, the solution $\nu^{\mathcal{U}}(c)$ is completely described by these relations:

$$\begin{cases} \nu^{\mathcal{U}}(c) = \left(M_{\mathcal{U}}^\top \left(\rho I - M_{\mathcal{W}}[d]^{-1} M_{\mathcal{W}}^\top\right)^{-1} M_{\mathcal{U}} \mathbb{1}, d\right) \\ \mathbb{1}^\top M_{\mathcal{U}}^\top \left(\rho I - M_{\mathcal{W}}[d]^{-1} M_{\mathcal{W}}^\top\right)^{-1} M_{\mathcal{U}} \mathbb{1} = c - \mathbb{1}^\top d. \end{cases} \tag{4}$$



Step 0:

$$c > c^0 \implies \nu^*(c) = \nu^0(c)\,.$$

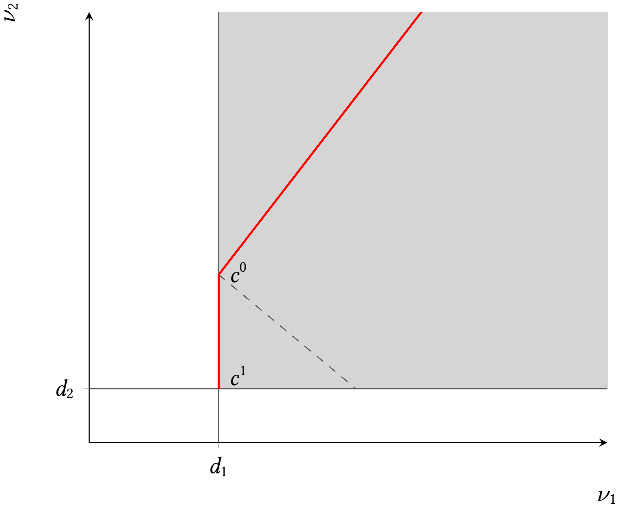


Step 1:

$$c = c^0 \implies \exists i \in \mathcal{S} \;\text{ s.t }\; \nu_i^*(c) = \pi_i c = d_i \,.$$

$$\text{define } \mathcal{W}^0 = \{i\},\ \mathcal{U}^0 = \mathcal{S}\backslash\{i\}\,,$$

$$\exists\ \mathbb{1}^\top d < c^1 < c^0 \text{ s.t } \quad \nu^*(c) = \min_{\nu \in \mathcal{Q}_c^{\mathcal{U}^0}} \phi(\nu)\,, \quad \text{for } \ c \in (c^1, c^0]\,.$$



Step 2 and so on: $c = c^1 \implies \exists j \in \mathcal{U}^0 \;$ s.t $\; \nu_j^*(c) = d_j \,.$

$$\text{define } \mathcal{W}^1 = \{i, j\}, \mathcal{U}^1 = \mathcal{S} \setminus \{i, j\} \,,$$

$$\exists \; \mathbb{1}^\top d < c^2 < c^1 \text{ s.t } \quad \nu^*(c) = \min_{\nu \in \mathcal{Q}_c^{\mathcal{U}^1}} \phi(\nu) \,, \qquad \text{for } c \in (c^2, c^1] \,.$$

Characterization of $\nu^*(c)$

Theorem

Fix $d \in \mathbb{R}^m_{++}$. Then,

- *the function $\nu^* : \mathcal{Q}_c \to \mathbb{R}^m_{++}$ is continuous and (entrywise) non decreasing;*
- *there exists a finite sequence of points $c^0 = \max_{i=1}^{m} \frac{d_i}{\pi_i} > c^1 > \cdots > c^s = \mathbb{1}'d$ and subsets $\mathcal{S} \supsetneq \mathcal{U}^0 \supsetneq \mathcal{U}^1 \supsetneq \cdots \supsetneq \mathcal{U}^{s-1}$ such that*

$$\mathcal{U}_c = \begin{cases} \mathcal{S} & \text{if } c > c^0 \\ \mathcal{U}^k & \text{if } c^{k+1} < c \leq c^k, \; k = 0, \ldots, s-1 \end{cases}$$

$$\nu^*(c) = \begin{cases} \nu^0(c) & \text{if } c > c^0 \\ \nu^{\mathcal{U}^k}(c) & \text{if } c^{k+1} < c \leq c^k, \; k = 0, \ldots, s-1. \end{cases}$$

Conclusions and future research

- iterative solution to calculate $\nu^*(c)$
- sensitivity of $\nu^*(c)$ in c
- new notion of centrality π

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- network with more components (delete **Assumption 2**)
- consider other system measures, e.g. **polarization**
- consider other resource constraints

Thank you for the attention

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