Some results on stopping games: mixed strategies and uncertain competition

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Outline

1. Introduction
2. Motivating example
3. A dynamic game

Introduction

Introduction I

Aims of the talk: To illustrate how uncertainty about competition radically changes optimal strategies in nonzero-sum Dynkin games. To demonstrate the importance of mixed strategies in complementing pure strategies.

The talk is based on

- De Angelis, Ekström (2020). *Playing with ghosts in a Dynkin game.* **Stoch. Process. Appl.** 130, pp. 6133-6156.

Introduction II

Related topics/literature:

- Standard Dynkin games with full information (broad body of existing literature: Dynkin, 1969, Bismut, 1977, Lepeltier and Maingueneau, 1984, Stettner, 1982, 1983, 1984, Yasuda, 1985, Kifer, 2000, Ekström and Peskir, 2008)
- Stoch. diff. games with asymmetric information (Cardaliaguet and Rainer, 2009)
- Dynkin games with asymmetric information (Grün, 2013, Gensbittel and Grün, 2019)
- Radomised stopping times as increasing processes (Baxter and Chacon, 1977, Meyer, 1978, Touzi and Vieille, 2002)
- Auction theory
- Basic filtering

Introduction III

More games with uncertain competition:

- Ekström, Lindensjö, Olofsson (2022). *How to detect a salami slicer: a stochastic controller-and-stopper game with unknown competition.* **SIAM J. Control Optim.**, 60(1), 545-574
- Ekström, Milazzo, Olofsson (2022). *The De Finetti problem with unknown competition.* **arXiv**:2204.07016

Introduction IV

Dynkin games with partial/asymmetric information:

- De Angelis, Merkulov, Palczewski (2022). *On the value of non-Markovian Dynkin games with partial and asymmetric information.* **Ann. Appl. Probab.** 32 (3), pp. 1774-1813
- De Angelis, Ekström, Glover (2022). *Dynkin games with incomplete and asymmetric information.* **Math. Oper. Res.** 47 (1), pp. 560-586
- De Angelis, Gensbttel, Villeneuve (2021). *A Dynkin game on assets with incomplete information on the return.* **Math. Oper. Res.** 46, (1), pp. 28-60

Motivating example:

A static game (Erik's example)

Sealed-bid auction with known competition:

- Two players bid for a good worth 1 EUR
- The bids are not public
- Both players know there are two bids (N bids)
- P1 bids $s \in [0,1]$ and P2 bids $t \in [0,1]$
- Payoffs:

$$\mathcal{J}_1(s,t) = (1-s)1_{\{s>t\}} \quad \text{and} \quad \mathcal{J}_2(s,t) = (1-t)1_{\{t>s\}}$$

- The only equilibrium is $(s_*, t_*) = (1,1)$ with $\mathcal{J}_1^* = \mathcal{J}_2^* = 0$

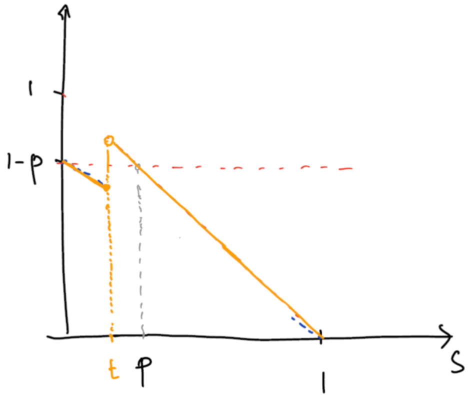
Sealed-bid auction with unknown competition (pure strategies):

- Same setup as above but now players are not sure whether there is another bidder
- For simplicity we take symmetric game
- P1 estimates that P2 is in the game (and viceversa) with probability $p \in (0,1)$
- Expected payoffs:

$$\mathcal{J}_1(s,t) = p(1-s)1_{\{s>t\}} + (1-p)(1-s) \quad \text{and} \quad \mathcal{J}_2(s,t) = p(1-t)1_{\{t>s\}} + (1-p)(1-t)$$

- There is no equilibrium in pure strategies:
  - If P2 bids $t < p$, then $P1$'s best response is $s = t + \varepsilon$ for $\varepsilon \downarrow 0$ (and viceversa)
  - If P2 bids $t > p$, then $P1$'s best response is $s = 0$ (and viceversa)
  - Players preempt each other for as long as they bid below p
  - The pair (p,p) is not an equilibrium

Figure: An illustration of Player 1′s payoff when Player 2 picks $t \in [0, p)$.



Sealed-bid auction with unknown competition (mixed strategies):

- Players use mixed strategies, i.e., their bid is drawn from a cdf F supported on $[0, p]$ with $F(0) = 0$
- If P2 bids according to F, then

$$\mathcal{J}_1(s, F) = p(1-s)F(s) + (1-p)(1-s)$$

- In equilibrium P1 is indifferent across $s \in [0, p]$, i.e. $\mathcal{J}_1(s, F) = const.$
- In particular, $\mathcal{J}_1(s, F) = \mathcal{J}_1(0, F) = (1-p)$ for all $s \in [0, p]$. It follows:

$$F(s) = \left(\frac{1-p}{p}\right)\frac{s}{1-s}, \; s \in [0, p] \; and \; F(s) = 1, \; s \in (p, 1].$$

- Notice that for $s \in (p, 1]$, $\mathcal{J}_1(s, F) = (1-s) < 1-p \implies$ no bid above p
- Equilibrium in mixed strategies $(s, t) \sim (F, F)$ and $\mathcal{J}_1(F, F) = \mathcal{J}_2(F, F) = 1-p$

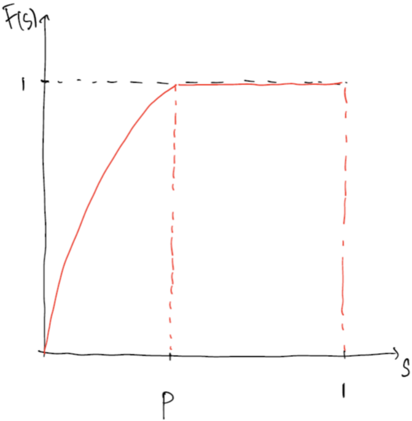


Figure: An illustration of the optimal mixed strategy.

A dynamic game:

Uncertain Competition

Probabilistic setup:
Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space hosting the following:

(a) a continuous, $\mathbb{R}^d$-valued, strong Markov process X which is regular (it can reach any open set in finite time with positive probability, for any value of the initial point $X_0 = x$)

(b) two Bernoulli distributed random variables θ_i, $i = 1, 2$

(c) two Unif$(0,1)$-distributed random variables U_i, $i = 1, 2$

Furthermore, we assume that these processes and random variables are mutually independent, and that $\mathbf{P}(\theta_i = 1) = 1 - \mathbf{P}(\theta_i = 0) = p_i \in (0, 1]$.

Remark: It is sometimes convenient to think of

$$(\Omega, \mathcal{F}) = (\Omega' \times [0,1]^2 \times \{0,1\}^2, \mathcal{F}' \times \mathcal{B}([0,1]^2) \times \mathcal{P}(\{0,1\}^2))$$

and $\mathbf{P} = \mathbf{P}' \times \mathcal{L}eb([0,1]^2) \times (p_1\delta_1 + (1-p_1)\delta_0) \times (p_2\delta_1 + (1-p_2)\delta_0)$

What players observe:

- There are two players in the game
- Both players observe the dynamics of X
- Players know the payoff of the game
- The i-th player does not observe directly θ_i, i.e., they don't know if they have competition
- The random variables U_i, $= 1, 2$, are randomisation devices. Each player observes at most her own U_i

Pure and mixed stopping times:

- The observation of X corresponds to knowledge of the filtration $\mathcal{F}_t^X = \sigma(X_s, 0 \le s \le t)$, $t \ge 0$
- A $\mathcal{F}_t^X$-stopping time (pure), $\tau \in \mathcal{T}$, is a $\mathcal{F}$-measurable mapping $\omega \mapsto \tau(\omega)$ s.t.

$$\{\tau \le t\} \in \mathcal{F}_t^X, \; \forall t \ge 0$$

- A randomised stopping time (mixed), $\tau \in \mathcal{T}^R$, is constructed as follows:
  - Product space $(\Omega, \mathcal{F}, \mathbf{P}) = (\Omega' \times [0,1], \mathcal{F}' \times \mathcal{B}([0,1]), \mathbf{P}' \times \mathcal{L}eb)$
  - $\omega \in \Omega \iff \omega = (\omega', u)$ with $\omega' \in \mathcal{F}'$ and $u \in [0,1]$
  - Let $U(\omega) = u$ be the randomisation device. Notice that, under $\mathbf{P}$, $U \sim \text{Unif}(0,1)$ and it is independent of $\mathcal{F}'$ (hence of $\mathcal{F}_\infty^X \subseteq \mathcal{F}'$)
  - Take a $\mathcal{F}$-measurable mapping $\tau : \Omega \to [0, \infty)$ such that $\omega' \mapsto \tau(\omega', u)$ is a $\mathcal{F}_t^X$-stopping time for each $u \in [0,1]$
  - Notice that $\tau(\omega) = \tau(\omega', U(\omega))$

Game's structure:

Notation:

- $\{\theta_i = 1\} \iff$ active competition for the i-th player
- For $i = 1, 2$, $\tau \in \mathcal{T}_i^R \iff \tau$ is randomised with randomisation device U_i

A preemption game with uncertain competition:

- The payoff: $g : \mathbb{R}^d \to [0, \infty)$ is a continuous function such that $\sup_{x \in \mathbb{R}^d} g(x) > 0$
- Player 1 chooses $\tau \in \mathcal{T}_1^R$ and Player 2 chooses $\gamma \in \mathcal{T}_2^R$
- The payoff for Player 1 at time τ is

$$R(\tau, \gamma) := \left(g(X_\tau) 1_{\{\tau < \hat{\gamma}\}} + \frac{1}{2} g(X_\tau) 1_{\{\tau = \hat{\gamma}\}} \right) 1_{\{\tau < \infty\}},$$

  where $\hat{\gamma} := \gamma 1_{\{\theta_1 = 1\}} + \infty 1_{\{\theta_1 = 0\}}$
- For Player 2 at time γ the payoff is $R(\gamma, \tau)$ with $\hat{\tau} := \tau 1_{\{\theta_2 = 1\}} + \infty 1_{\{\theta_2 = 0\}}$
- Both players are maximisers (of the expected future payoff)

Equilibrium in the game:

Denote $\mathcal{J}_1(\tau,\gamma;p_1,x) := \mathsf{E}_x[R(\tau,\gamma)]$ and $\mathcal{J}_2(\tau,\gamma;p_2,x) := \mathsf{E}_x[R(\gamma,\tau)]$.

Definition (Nash equilibrium). Given $x \in \mathbb{R}^d$ and $p_i \in (0,1]$, $i = 1,2$, a pair $(\tau^*,\gamma^*) \in \mathcal{T}_1^R \times \mathcal{T}_2^R$ is a Nash equilibrium if

$$\mathcal{J}_1(\tau,\gamma^*;p_1,x) \le \mathcal{J}_1(\tau^*,\gamma^*;p_1,x)$$

and

$$\mathcal{J}_2(\tau^*,\gamma;p_2,x) \le \mathcal{J}_2(\tau^*,\gamma^*;p_2,x)$$

for all pairs $(\tau,\gamma) \in \mathcal{T}_1^R \times \mathcal{T}_2^R$. Given an equilibrium pair $(\tau^*,\gamma^*) \in \mathcal{T}_1^R \times \mathcal{T}_2^R$ we define the equilibrium payoffs as

$$v_i(p_i,x) := \mathcal{J}_i(\tau^*,\gamma^*;p_i,x), \qquad \text{for } i = 1,2.$$

Some preliminary considerations:

- Value of the single agent problem

$$V(x) := \sup_{\tau} \mathbf{E}_x \left[e^{-r\tau} g(X_\tau) 1_{\{\tau<\infty\}} \right]$$

- $\tau^*_V := \inf\{t \geq 0 : V(X_t) = g(X_t)\}$ is optimal for V
- In the 2-player game, if P1 chooses τ^*_V, they receive

$$\mathcal{J}_1(\tau^*_V, \gamma; p_1, x) \geq (1 - p_1)V(x) \quad (\textit{safety value})$$

  for any $\gamma \in \mathcal{T}_2^R$ (analogously for P2)

- Assumptions:
  - $V \in C(\mathbb{R}^d)$
  - $\mathbf{E}_x \left[\sup_{t\geq 0} e^{-rt} g(X_t) \right] < \infty$, $x \in \mathbb{R}^d$
  - $\limsup_{t\to\infty} e^{-rt} V(X_t) 1_{\{\tau^*_V=+\infty\}} = 0$

An observation:
Letting $\tau \in \mathcal{T}_1^R$ and $\gamma \in \mathcal{T}_2^R$ be arbitrary one has

$$\sup_{\zeta \in \mathcal{T}_1^R} \mathcal{J}_1(\zeta, \gamma; p_1, x) = \sup_{\zeta \in \mathcal{T}} \mathcal{J}_1(\zeta, \gamma; p_1, x) \quad \textit{and} \quad \sup_{\zeta \in \mathcal{T}_2^R} \mathcal{J}_2(\tau, \zeta; p_2, x) = \sup_{\zeta \in \mathcal{T}} \mathcal{J}_2(\tau, \zeta; p_2, x).$$

Proof: For $\zeta \in \mathcal{T}_1^R$ we denote $\zeta = \zeta(u)$ conditional upon $U_1 = u$. Then

$$\sup_{\zeta \in \mathcal{T}} \mathcal{J}_1(\zeta, \gamma; p_1, x) \le \sup_{\zeta \in \mathcal{T}_1^R} \mathcal{J}_1(\zeta, \gamma; p_1, x)$$

$$= \sup_{\zeta \in \mathcal{T}_1^R} \int_0^1 \mathcal{J}_1(\zeta(u), \gamma; p_1, x) \mathrm{d}u \le \sup_{\nu \in \mathcal{T}} \mathcal{J}_1(\nu, \gamma; p_1, x).$$

That is, randomisation does not increase the payoff and it is only needed to find an equilibrium.

Notation: $\Gamma \in \mathcal{A}$ iff

- Γ is right-continuous, non-decreasing, $\mathcal{F}^X$-adapted processes
- $\Gamma_{0-} = 0$ and $\Gamma_t \leq 1$ for all $t \geq 0$.

An equivalent form of randomised stopping times:
[Bismut, Baxter, Chacon, Meyer]

Let $U \sim \text{Unif}(0,1)$, independent of X, be a randomisation device for $\gamma \in \mathcal{T}^R$. Then

$$\gamma = \inf\{t \geq 0 : \Gamma_t > U\}$$

for some $\Gamma \in \mathcal{A}$. Furthermore, we say that γ is *generated* by Γ.

A representation of the payoffs:

Let $\tau \in \mathcal{T}_1^R$ and $\gamma \in \mathcal{T}_2^R$ be generated by Γ^1 and Γ^2 in $\mathcal{A}$, respectively. For any $\zeta \in \mathcal{T}$ and $x \in \mathbb{R}^d$ we have

$$\begin{aligned}\mathcal{J}_1(\zeta,\gamma;p_1,x) =&(1-p_1)\mathsf{E}_x\Big[e^{-r\zeta}g(X_\zeta)1_{\{\zeta<+\infty\}}\Big]\\ &+p_1\mathsf{E}_x\Big[e^{-r\zeta}g(X_\zeta)(1-\Gamma^2_\zeta)1_{\{\zeta<+\infty\}}+e^{-r\zeta}\tfrac{1}{2}g(X_\zeta)\Delta\Gamma^2_\zeta 1_{\{\zeta<+\infty\}}\Big]\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_2(\tau,\zeta;p_2,x) =&(1-p_2)\mathsf{E}_x\Big[e^{-r\zeta}g(X_\zeta)1_{\{\zeta<+\infty\}}\Big]\\ &+p_2\mathsf{E}_x\Big[e^{-r\zeta}g(X_\zeta)(1-\Gamma^1_\zeta)1_{\{\zeta<+\infty\}}+e^{-r\zeta}\tfrac{1}{2}g(X_\zeta)\Delta\Gamma^1_\zeta 1_{\{\zeta<+\infty\}}\Big].\end{aligned}$$

The belief processes:

If $\gamma \in \mathcal{T}_2^R$ is generated by $\Gamma^2 \in \mathcal{A}$, then Player 1 dynamically evaluates the conditional probability of Player 2 being active as

$$\begin{aligned}\Pi_t^1 :=& \mathbf{P}(\theta_1 = 1 | \mathcal{F}_t^X, \hat{\gamma} > t) \\ =& \frac{\mathbf{P}(\theta_1 = 1 | \mathcal{F}_t^X) \mathbf{P}(\hat{\gamma} > t | \mathcal{F}_t^X, \theta_1 = 1)}{\mathbf{P}(\hat{\gamma} > t | \mathcal{F}_t^X)} \\ =& \frac{p_1 \mathbf{P}(\gamma > t | \mathcal{F}_t^X)}{1 - p_1 + p_1 \mathbf{P}(\gamma > t | \mathcal{F}_t^X)} = \frac{p_1 (1 - \Gamma_t^2)}{1 - p_1 \Gamma_t^2}\end{aligned}$$

provided $p_1 \in (0,1)$. Likewise, if $\tau \in \mathcal{T}_1^R$ is generated by $\Gamma^1 \in \mathcal{A}$, then

$$\Pi_t^2 := \mathbf{P}(\theta_2 = 1 | \mathcal{F}_t^X, \hat{\tau} > t) = \frac{p_2 (1 - \Gamma_t^1)}{1 - p_2 \Gamma_t^1}$$

provided $p_2 \in (0,1)$.

A one-to-one correspondence:

There is a one-to-one correspondence between Γ^2 and Π^1 (analogous for Γ^1 and Π^2). In fact,

$$\Gamma_t^2 = \frac{p_1 - \Pi_t^1}{p_1(1 - \Pi_t^1)}.$$

Remark: If P2 wants to generate a certain belief Π^1 of P1's, then she must construct Γ^2 as above (analogously swapping the roles of $P1$ and $P2$).

Three sets:

Recall V for the single agent problem. Equilibria in the game are fully determined in terms of three sets

$$\overline{\mathcal{C}} := \{(p,x) \in (0,1) \times \mathbb{R}^d : (1-p)V(x) \geq g(x)\}$$

$$\mathcal{C}' := \{(p,x) \in (0,1) \times \mathbb{R}^d : (1-p/2)g(x) < (1-p)V(x) < g(x)\}$$

$$\mathcal{S} := \{(p,x) \in (0,1) \times \mathbb{R}^d : (1-p)V(x) \leq (1-p/2)g(x)\}$$

and note that $\overline{\mathcal{C}} \cup \mathcal{C}' \cup \mathcal{S} = (0,1) \times \mathbb{R}^d$.

The explicit boundaries:

It is easy to see that

$$\overline{\mathcal{C}} = \{(p,x) \in (0,1) \times \mathbb{R}^d : p \leq b(x)\},$$
$$\mathcal{C}' = \{(p,x) \in (0,1) \times \mathbb{R}^d : b(x) < p < c(x),\}$$
$$\mathcal{S} = \{(p,x) \in (0,1) \times \mathbb{R}^d : c(x) \leq p\},$$

with continuous boundaries $b \leq c$ given by

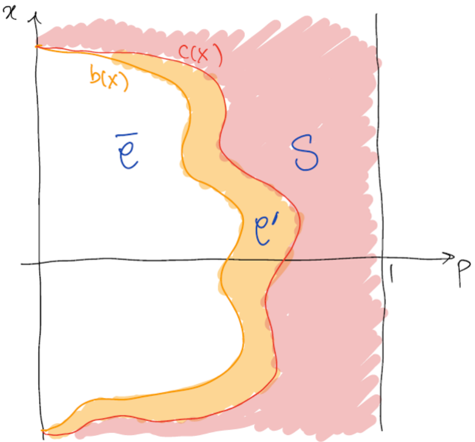
$$b(x) = 1 - \frac{g(x)}{V(x)} \qquad \text{and} \qquad c(x) = \frac{V(x) - g(x)}{V(x) - g(x)/2}.$$

Moreover,

(i) $b(x) = c(x) = 0 \iff V(x) = g(x)$;

(ii) $b(x) = c(x) = 1 \iff g(x) = 0$.

Figure: An illustration of the sets $\overline{\mathcal{C}}, \mathcal{C}'$ and $\mathcal{S}$.



A reflecting (belief) process:

Proposition. Let $(p,x)\in\overline{\mathcal{C}}$ be given and fixed and define P_x-a.s. the process

$$Z_t := p \wedge \inf_{0\le s\le t} b(X_s). \tag{1}$$

Then P_x-a.s.

(i) Z is non-increasing and continuous;

(ii) $(Z_t, X_t)\in\overline{\mathcal{C}}$ for all $t\ge 0$;

(iii) we have

$$\mathrm{d}Z_t = 1_{\{(1-Z_t)V(X_t)=g(X_t)\}}\mathrm{d}Z_t \tag{2}$$

as (random) measures.

Remark: $Z=\Pi^\Gamma$ is the belief generated by

$$\Gamma_t := \frac{p - p\wedge \inf_{0\le s\le t} b(X_s)}{p\Big(1 - p\wedge \inf_{0\le s\le t} b(X_s)\Big)}, \quad t\ge 0,$$

and (Π^Γ, X) is kept inside $\overline{\mathcal{C}}$ with minimal effort.

Construction of Nash equilibria I:

From now on we assume $0 < p_1 \le p_2 < 1$. It turns out that in this setting Player 2 is the most active.

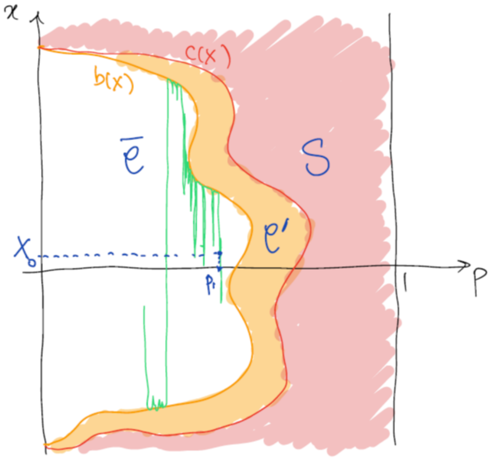
Equilibrium (part 1). If $(p_1, x) \in \mathcal{S}$, an equilibrium is for both players to stop at once.

Equilibrium (part 2). If $(p_1, x) \in \overline{\mathcal{C}}$,

- Player 2 picks $\Gamma^{2,*} \in \mathcal{A}$ such that the process $(\Pi^1_t, X_t)_{t\ge0}$ is kept in $\overline{\mathcal{C}}$ with minimal effort (recall $\Pi^1_t = p_1(1-\Gamma^2_t)/(1-p_1\Gamma^2_t)$).
- Player 1 picks

$$\Gamma^{1,*}_t := \frac{p_1}{p_2}\Gamma^{2,*}_t 1_{\{t<\tau^*_V\}} + 1_{\{t\ge\tau^*_V\}}.$$

Figure: An illustration of the pair (Π^1, X) associated to $\Gamma^{2,*}$ when $(p_1, x) \in \overline{\mathcal{C}}$.



Construction of Nash equilibria II:

Equilibrium (part 3). If $(p_1, x) \in \mathcal{C}'$,

- Player 2 picks $\Gamma^{2,*} \in \mathcal{A}$ such that the process $(\Pi^1_t, X_t)_{t\geq 0}$ makes an immediate jump to a point (q_1, x) with $q_1 < b(x)$. Then $(\Pi^1_t, X_t)_{t\geq 0}$ is kept in $\overline{\mathcal{C}}$ with minimal effort. (Note: We have an explicit expression for q_1 depending on $p_1, V(x)$ and $g(x)$.)
- Player 1 picks

$$\Gamma^{1,*}_t := \frac{p_1}{p_2}\Gamma^{2,*}_t 1_{\{t<\tau^*_V\}} + 1_{\{t\geq\tau^*_V\}}.$$

Remark. The jump of $\Gamma^{2,*}$ corresponds to saying that Player 2 'flicks a (biased) coin' and stops immediately with probability $\Gamma^{2,*}_0$ (known explicitly) or continues with probability $1 - \Gamma^{2,*}_0$.

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Construction of Nash equilibria II:

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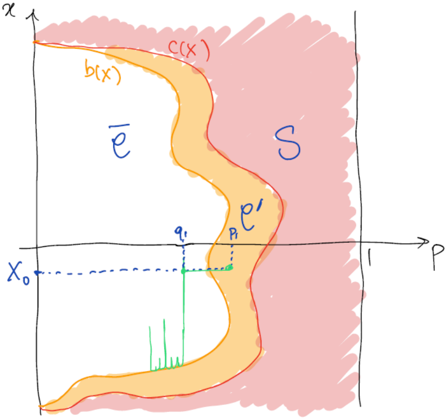
- Player 1 picks

$$\Gamma^{1,*}_t := \frac{p_1}{p_2} \Gamma^{2,*}_t 1_{\{t < \tau^*_V\}} + 1_{\{t \geq \tau^*_V\}}.$$

Remark. The jump of $\Gamma^{2,*}$ corresponds to saying that Player 2 'flicks a (biased) coin' and stops immediately with probability $\Gamma^{2,*}_0$ (known explicitly) or continues with probability $1 - \Gamma^{2,*}_0$.

Figure: An illustration of the pair (Π^1, X) associated to $\Gamma^{2,*}$ when $(p_1, x) \in \mathcal{C}'$.



Outline of proof:

(*Step 1*) For $(p_1, x) \in \overline{\mathcal{C}} \cup \mathcal{C}'$ set

$$\Gamma_0^* := \frac{2}{p_1}\left(1 - \frac{(1-p_1)V(x)}{g(x)}\right)^+ \quad \text{and} \quad q_1 := \frac{p_1(1-\Gamma_0^*)}{1-p_1\Gamma_0^*}$$

and note that

$$p_1 \le b(x) \implies \Gamma_0^* = 0 \textit{ and } q_1 = p_1 \in (0, b(x)] \ \ (\textit{no jump})$$

$$p_1 \in (b(x), c(x)) \implies \Gamma_0^* \in (0,1) \textit{ and } q_1 \in (0, b(x)) \ \ (\textit{jump to interior of } \mathcal{C})$$

We consider the process

$$N_t := \left(1 - \frac{p_1}{2}\Gamma_0^*\right) g(x) 1_{\{t=0\}} + \tilde{N}_t 1_{\{t>0\}}$$

where

$$\tilde{N}_t := (1_{\{\theta_1=0\}} + 1_{\{\theta_1=1, U_2 \ge \Gamma_0^*\}})(1-q_1)e^{-rt}V(X_t)$$

and show that

$$\sup_{\tau \in \mathcal{T}} \mathsf{E}_x\left[N_\tau\right] = (1-p_1)V(x),$$

with martingale methods.

(*Step 2*) Let γ^* be generated by $\Gamma^{2,*}$. We show that

$$\sup_{\tau\in\mathcal{T}_1^R} \mathcal{J}_1(\tau,\gamma^*;p_1,x) \le \sup_{\tau\in\mathcal{T}} \mathsf{E}_x\Big[N_\tau\Big]$$

and that, choosing τ^* generated by $\Gamma^{1,*}$ we obtain

$$\mathcal{J}_1(\tau^*,\gamma^*;p_1,x) = \sup_{\tau\in\mathcal{T}} \mathsf{E}_x\Big[N_\tau\Big].$$

Hence $\mathcal{J}_1(\tau^*,\gamma^*;p_1,x) = (1-p_1)V(x)$.

(*Step 3*) Finally, we show that

$$\sup_{\gamma \in \mathcal{T}_2^R} \mathcal{J}_2(\tau^*, \gamma; p_2, x) = \sup_{\tau \in \mathcal{T}} \mathbf{E}_x\Big[N_\tau\Big]$$

and that, choosing γ^* generated by $\Gamma^{2,*}$ we obtain

$$\mathcal{J}_2(\tau^*, \gamma^*; p_2, x) = \sup_{\tau \in \mathcal{T}} \mathbf{E}_x\Big[N_\tau\Big].$$

Hence $\mathcal{J}_2(\tau^*, \gamma^*; p_2, x) = (1 - p_1)V(x)$.

Steps 2 and 3 are accomplished using the formulae for $\mathcal{J}_i$ involving Γ^i.

□

Remark: Notice that $\mathcal{J}_1(\tau^*, \gamma^*; p_1, x) = \mathcal{J}_2(\tau^*, \gamma^*; p_2, x) = (1 - p_1)V(x)$.
That is, P1 scores just the safety value whereas P2 scores $(p_2 - p_1)V(x)$ above the safety value.

An example with explicit solution: Competing for a real option.

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t \quad \text{and} \quad g(x) = (x-K)^+ \ \textit{with } \mu < r.$$

For simplicity $p_1 = p_2 = p \in (0,1)$.

Value of the American call option (single-agent):

$$V(x) = \begin{cases} (B-K)(x/B)^\eta, & \text{for } x \in (0,B), \\ g(x), & \text{for } x \in [B,\infty), \end{cases}$$

where

$$\eta = \frac{\sigma^2 - 2\mu}{2\sigma^2} + \sqrt{\left(\frac{\sigma^2-2\mu}{2\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \in (1,\infty)$$

and $B := \eta K/(\eta - 1)$.

Since V and B are explicit, then we obtain

$$b(x) = 1 - \frac{g(x)}{V(x)} = \begin{cases} 1 & x \in (0, K] \\ 1 - \frac{(x-K)(B/x)^{\eta}}{B-K} & x \in (K, B) \\ 0 & x \in [B, \infty) \end{cases}$$

and

$$c(x) = \frac{V(x) - g(x)}{V(x) - g(x)/2} = \begin{cases} 1 & x \in (0, K] \\ 1 - \frac{(x-K)}{2(B-K)(x/B)^{\eta} - x + K} & x \in (K, B) \\ 0 & x \in [B, \infty). \end{cases}$$

Notice that

- $g(x) = 0$ and $b(x) = c(x) = 1$ for $x \leq K$
- $V(x) = g(x)$ and $b(x) = c(x) = 0$ for $x \geq B$

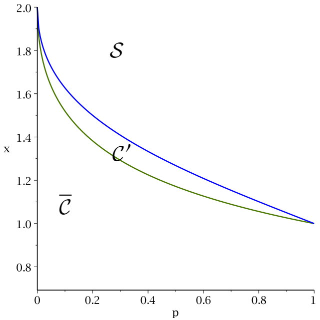


Figure: The figure displays the curves $p = b(x)$ (bottom) and $p = c(x)$ (top).

Conclusions

- Formulation of stochastic (Dynkin) nonzero-sum games with uncertain competition
- Need for mixed strategies
- Continuous time Markovian dynamics
- Bayesian evaluation of players′ belief
- Explicit construction of equilibria in Markovian setting

Some open questions

- Consolation prize for second mover
- More than 2 players
- Asymmetric payoffs across players

Thank you