Prediction Equilibria in Dynamic Traffic Assignment

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Passau University

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Content

- Part I: Basic DTA Theory
- Part II: Prediction Models in DTA

Part I

Basic DTA Theory

Layered DTA Modelling

DTA as a two-layer system:

- physical layer
  - flow propagation (discrete, continuous)
  - queueing models (Vickrey point queue, horizontal queue,)
  - outflow functions (depending on travel time, volume, FIFO)
  - travel time functions (depending on inflow, flow volume, FIFO)
  - ...

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  - ...
- behavioral layer
  - How do/can travelers behave?
    - fixed departure rate, flexible departure rate, elastic demand,..
  - What is the information available?
    - full information, partial information, prediction,..

Research Questions

Given a physical flow model and a behavioral model:

- Do dynamic equilibria exist ?
- How can they be computed ?
- Are they unique ?
- Does the equilibrium set have a specific structure?
- Price of anarchy/stability ?
- Engineer equilibria (tolls, network design, *information design*,...)?

Modelling dynamic equilibria ("Terry Friesz school"):

- full information
  - strategy space $\Lambda_i := \{h \in (L_2[0,T])^{\mathcal{P}_i} \,|\, \sum_{p \in \mathcal{P}_i} h_p(t) = r_i(t), t \in [0,T]\}$ with $h_p(t)$ inflow rate into path p for commodity $i \in I$.

Mathematical Tools

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  - path-delay function (using $\mathcal{P} = \cup_{i \in I} \mathcal{P}_i$)

$$\Psi : (L_2[0,T])^{\mathcal{P}} \to (L_2[0,T])^{\mathcal{P}}$$
$$h(\cdot) \mapsto (\Psi_p(\cdot, h))_{p \in \mathcal{P}}.$$

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Definition 1

$h^* \in \Lambda$ is a dynamic equilibrium with fixed inflow rates, if for all $i \in I$, the following conditions hold:

$$h_p^*(t) > 0, p \in \mathcal{P}_i \Rightarrow \Psi_p(t, h^*) \leq \Psi_q(t, h^*) \quad (1)$$
$$\text{for almost all } t \in [0,T], q \in \mathcal{P}_i. \quad (2)$$

Friesz et al. (1993,+), Zhu and Marcote (2000)

Theorem 2

Let $\Psi_p(\cdot, h)$ be positive and measurable for all $p \in \mathcal{P}$ and $h \in \Lambda$. Then, $h^* \in \Lambda$ is a dynamic equilibrium with fixed inflow rates r, if and only if it solves the following variational inequality:

$$\begin{aligned} &\textit{Find } h^* \in \Lambda \textit{ such that}: \\ &\qquad \langle \Psi(h^*), (h - h^*) \rangle \geq 0 \textit{ for all } h \in \Lambda. \end{aligned} \qquad (\mathrm{VI}(\Psi, r, [0, T]))$$

Existence

"Friesz School (starting from 2015)"

- Browder (1968): Λ convex and compact, Ψ "continuous" $\Rightarrow$ VI has solution h^*.
- Use piecewise constant h_p^i with discretization $\Delta_i \Rightarrow \Lambda(\Delta_i)$ compact.
- Take the limit $\Delta_i \to 0$ and prove convergence.

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"Zhu and Marcotte, Cominetti, Correa et al. School (starting from 2000, 2015)"

- Brézis, Lions (1968): Λ convex and closed and bounded, Ψ "sequentially weak-strong continuous" $\Rightarrow$ VI has solution h^*.
- Prove "sequential weak-strong continuity" of Ψ.

Computing Equilibria – "Friesz School"

Pose the problem as a fixed point "path-balancing" problem:

$$h_p^{k+1}(t) := \left[h_p^k(\theta) - \alpha^k \cdot \left(\Psi_p(t, h^k) - \min_q \{ \Psi_q(t, h^k) \} \right) \right]_+ .$$

Problem 1

- Convergence (local, global)?
- Numerical Stability ?
- Network Loading ?
- Relation to packet-based simulations (MatSim)?

“Koch/Skutella/Correa/Cominetti School”

Vickrey Model

- Use specific structure of “derivatives of equilibria”:
- Node label functions (specific form of $\Psi_p(t, h)$!)
- Thin flows
- “Extension methodology”

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Problem 2

- Finite number of extension steps ?
- For which physical models ($\Psi(t, h)$) can we apply the extension methodology ?

Uniqueness of Equilibria

uniqueness of dynamic equilibria in the Vickrey- and more general models, single sink!

- Cominetti et al. (2015)
- Smith and Iryo (2017)
- Olver et al. (2022)

Problem 3

For which physical models ($\Psi(t, h)$) can we say more about uniqueness ?

Part II

Information Models – Prediction Equilibria

Vickrey Queueing Model

- digraph $G = (V, E)$

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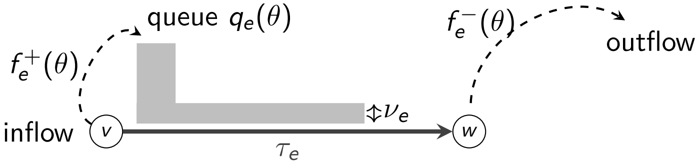

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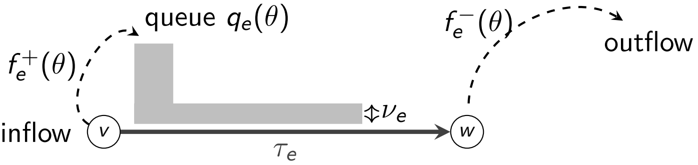

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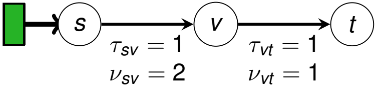

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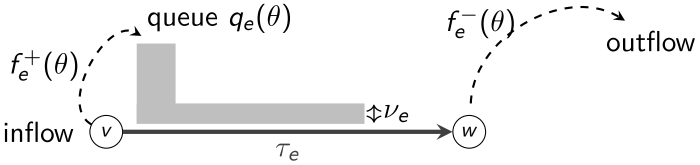

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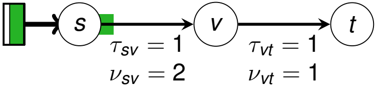

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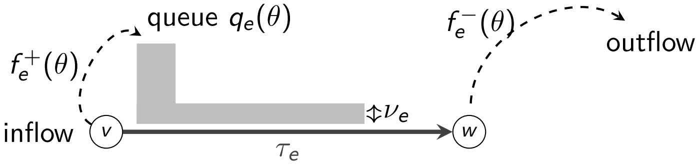

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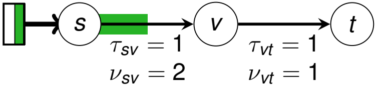

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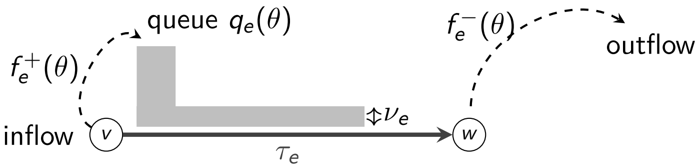

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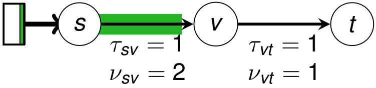

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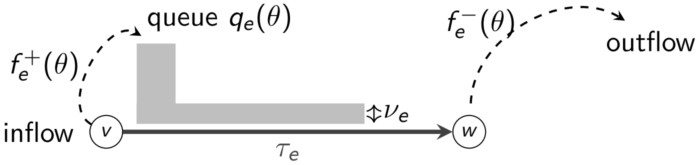

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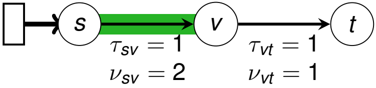

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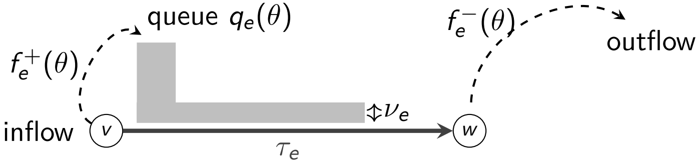

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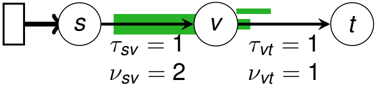

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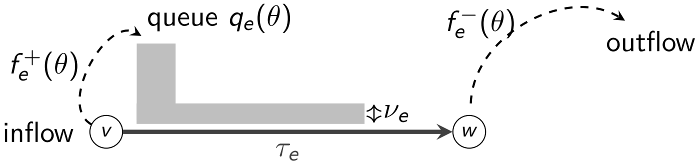

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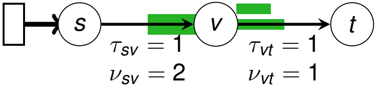

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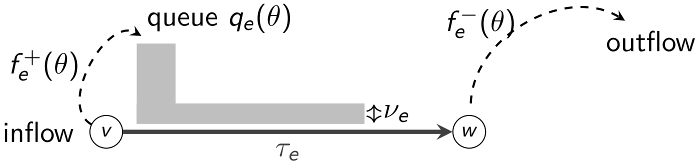

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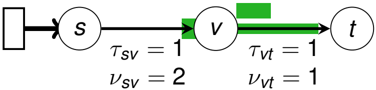

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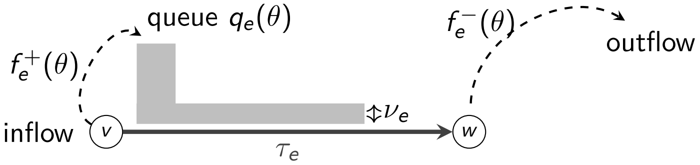

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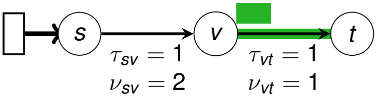
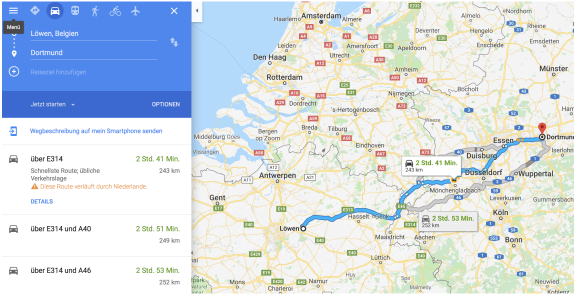


Route Planning



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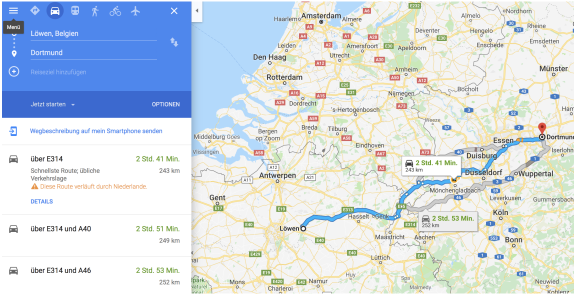

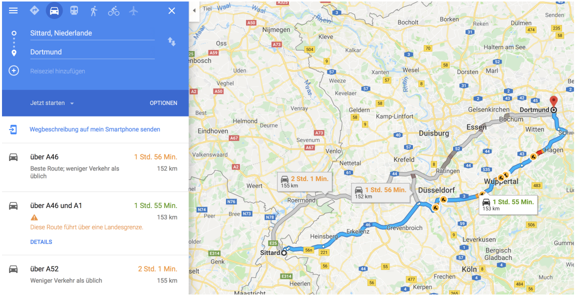

Route Planning

Current length of a v-t path P: travel time + waiting times in queues

$$c_P(\theta) = \sum_{e \in P} \tau_e + q_e(\theta)/\nu_e$$

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Definition 3 (Instantaneous Dynamic Equilibrium (IDE))

At every point in time: if positive flow enters an edge, the edge must lie on a <u>currently shortest path</u> towards the respective sink.

IDE – Single Sink

- *total travel time* of edge e at time θ: $c_e(\theta) = \tau_e + q_e(\theta)/\nu_e$
- define node labels $\ell_v(\theta)$ measuring the currently earliest arrival time at t:

$$\ell_v(\theta) = \begin{cases} \theta, & \text{for } v = t \\ \min_{e=vw\in E}\{c_e(\theta) + \ell_w(\theta)\}, & \text{for all } v \in V \setminus \{t\}. \end{cases}$$

Definition 4 (Active Edges)

An edge $e = vw \in E$ is active at time θ if

$$\ell_v(\theta) = \ell_w(\theta) + c_e(\theta).$$

$E(\theta) \subseteq E$ set of active edges

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Definition 5 (Instantaneous Dynamic Equilibrium)

For every $\theta \geq 0$: $f_e^+(\theta) > 0 \Rightarrow e \in E(\theta)$.

Example:

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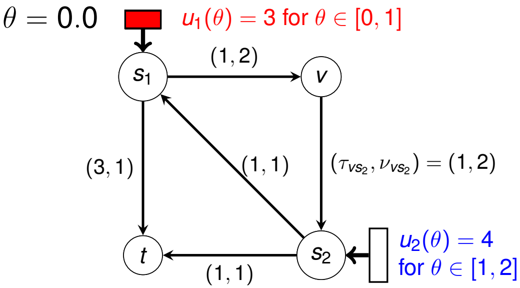
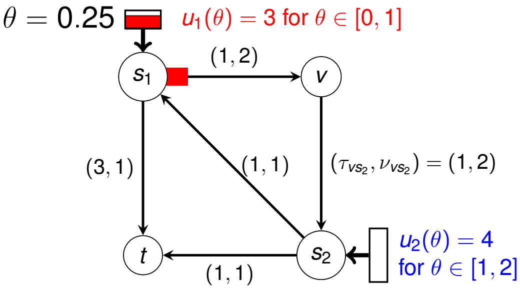


$\theta = 0.0$
$u_1(\theta) = 3$ for $\theta \in [0, 1]$
s_1
v
t
s_2
$(1, 2)$
$(3, 1)$
$(1, 1)$
$(\tau_{vs_2}, \nu_{vs_2}) = (1, 2)$
$(1, 1)$
$u_2(\theta) = 4$
for $\theta \in [1, 2]$

Example:



$\theta = 0.25$
$u_1(\theta) = 3$ for $\theta \in [0, 1]$
$(1, 2)$
s_1
v
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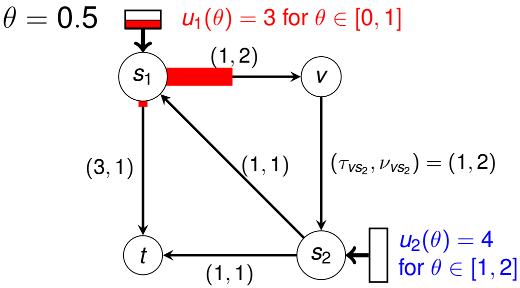

$\theta = 0.5$
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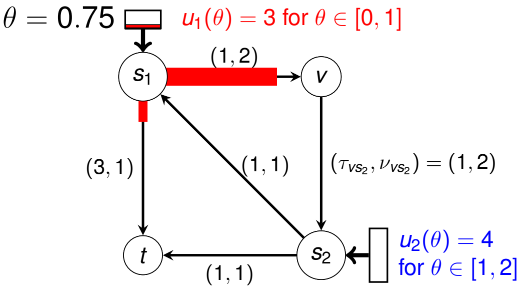

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Example:

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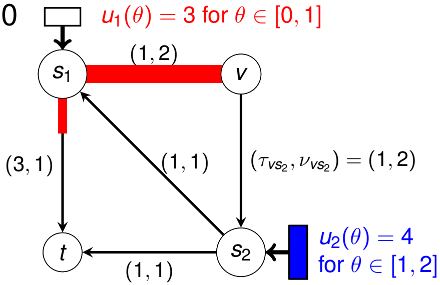

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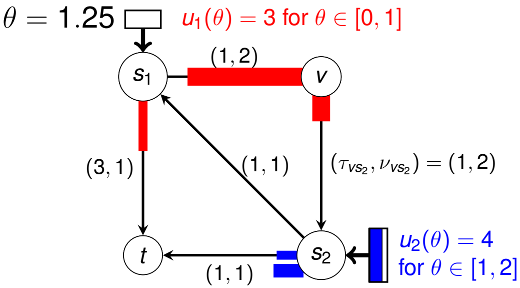
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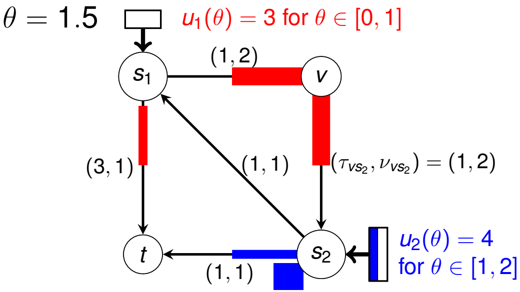

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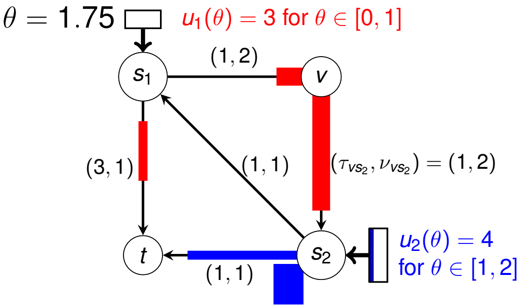

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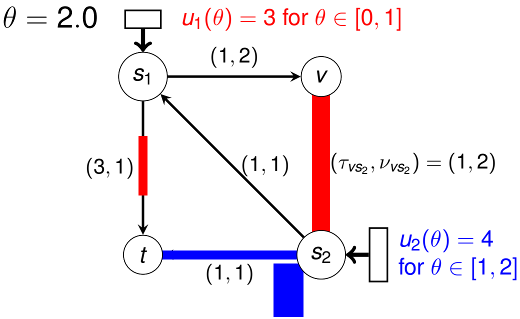
$u_2(\theta) = 4$
for $\theta \in [1, 2]$
$(1, 1)$

Example:


$\theta = 2.0$
$u_1(\theta) = 3$ for $\theta \in [0, 1]$
$(1, 2)$
s_1
v
$(3, 1)$
$(1, 1)$
$(\tau_{vs_2}, \nu_{vs_2}) = (1, 2)$
t
s_2
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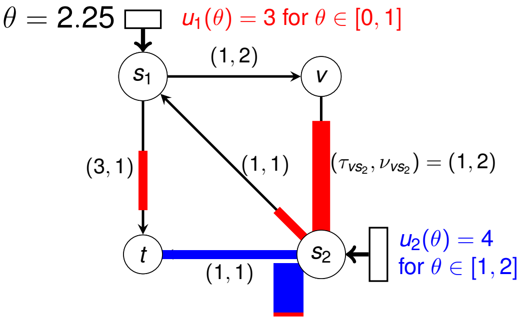

$\theta = 2.25$
$u_1(\theta) = 3$ for $\theta \in [0, 1]$
$(1, 2)$
s_1
v
$(3, 1)$
$(1, 1)$
$(\tau_{vs_2}, \nu_{vs_2}) = (1, 2)$
t
s_2
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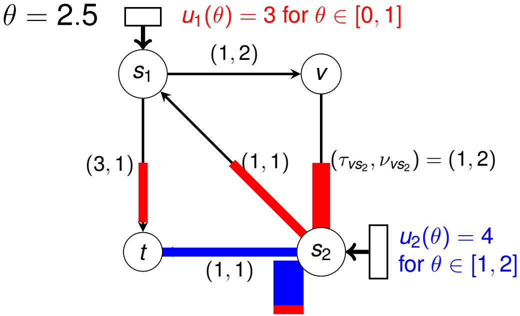
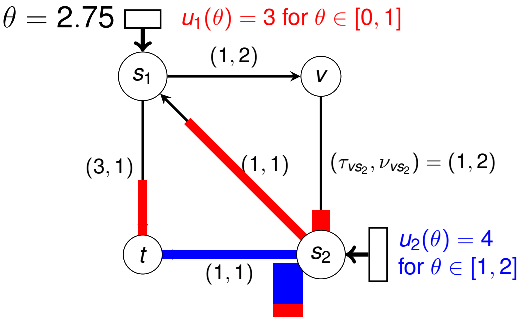


$\theta = 2.5$
$u_1(\theta) = 3$ for $\theta \in [0, 1]$
(1, 2)
s_1
v
(3, 1)
(1, 1)
$(\tau_{vs_2}, \nu_{vs_2}) = (1, 2)$
t
s_2
(1, 1)
$u_2(\theta) = 4$
for $\theta \in [1, 2]$

Example:



$\theta = 2.75$
$u_1(\theta) = 3$ for $\theta \in [0, 1]$
$(1, 2)$
s_1
v
$(3, 1)$
$(1, 1)$
$(\tau_{vs_2}, \nu_{vs_2}) = (1, 2)$
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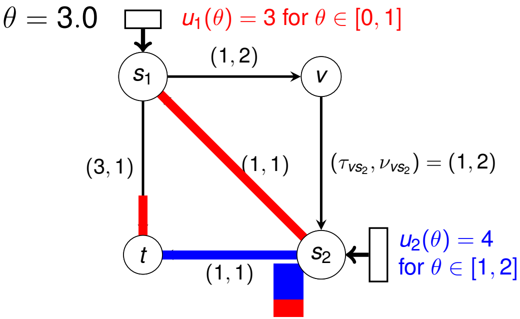
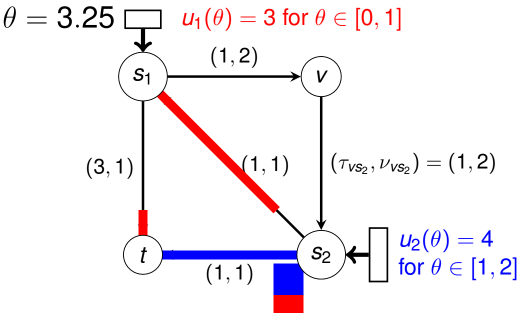


$\theta = 3.0$
$u_1(\theta) = 3$ for $\theta \in [0, 1]$
$(1, 2)$
s_1
v
$(3, 1)$
$(1, 1)$
$(\tau_{vs_2}, \nu_{vs_2}) = (1, 2)$
t
s_2
$(1, 1)$
$u_2(\theta) = 4$
for $\theta \in [1, 2]$

Example:



$\theta = 3.25$
$u_1(\theta) = 3$ for $\theta \in [0,1]$
$(1,2)$
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$(1,1)$
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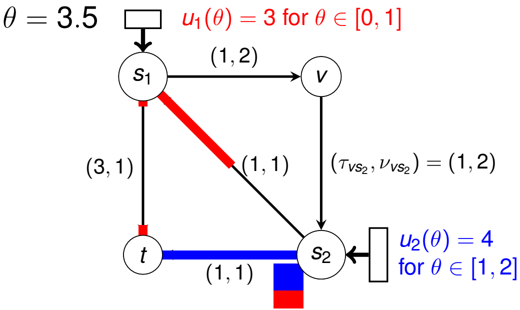

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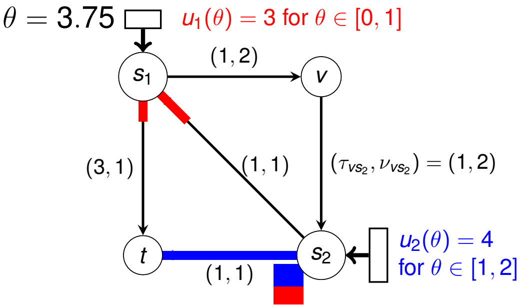

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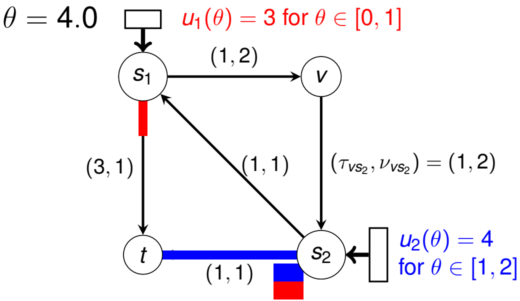
$\theta = 4.0$
$u_1(\theta) = 3$ for $\theta \in [0, 1]$
s_1
v
t
s_2
$(1, 2)$
$(3, 1)$
$(1, 1)$
$(\tau_{vs_2}, \nu_{vs_2}) = (1, 2)$
$(1, 1)$
$u_2(\theta) = 4$
for $\theta \in [1, 2]$

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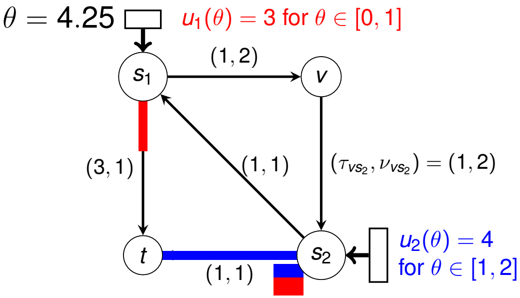
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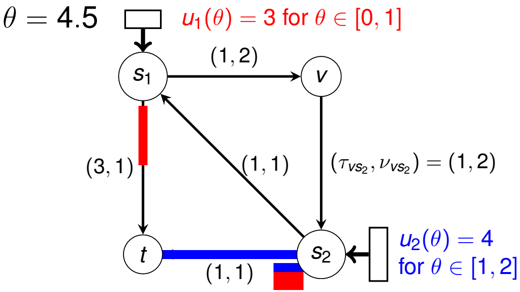

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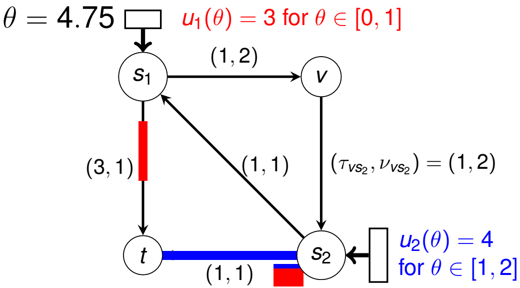
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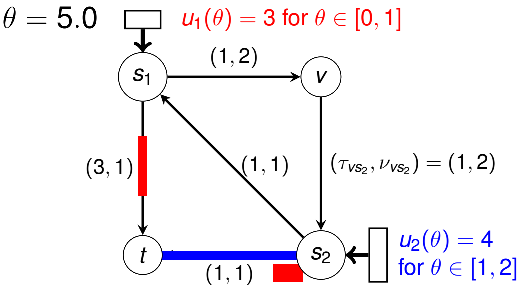

$\theta = 5.0$
$u_1(\theta) = 3$ for $\theta \in [0, 1]$
s_1
$(1, 2)$
v
$(3, 1)$
$(1, 1)$
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t
s_2
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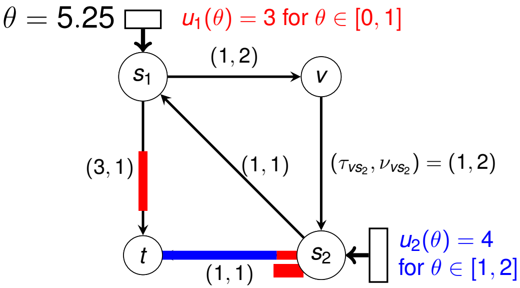

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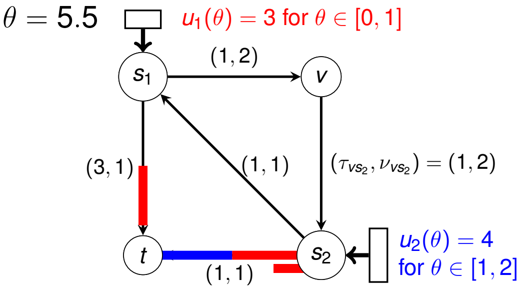

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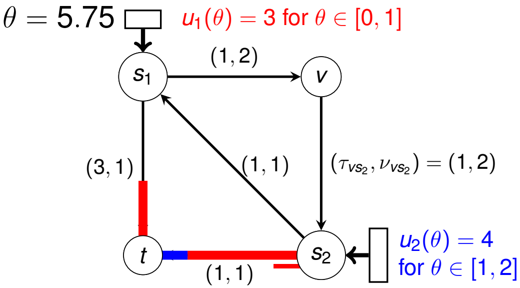

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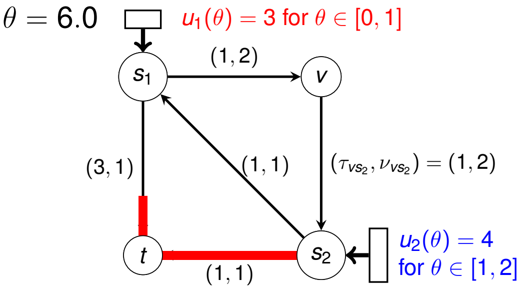
$\theta = 6.0$
$u_1(\theta) = 3$ for $\theta \in [0, 1]$
$(1, 2)$
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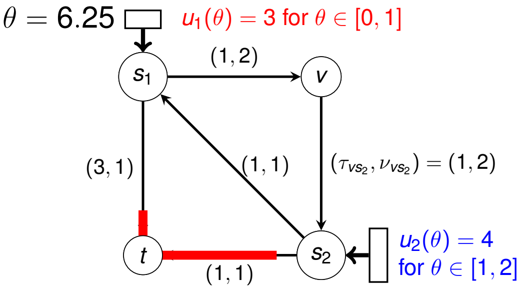
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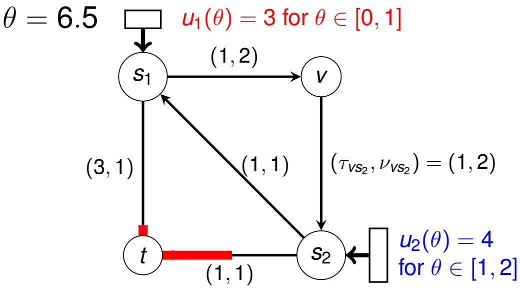
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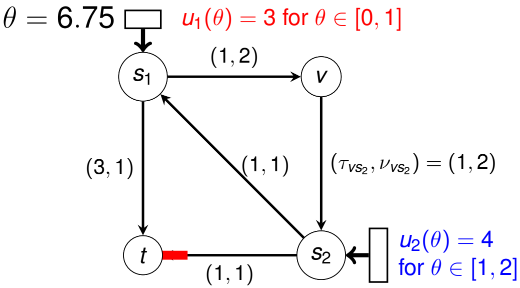
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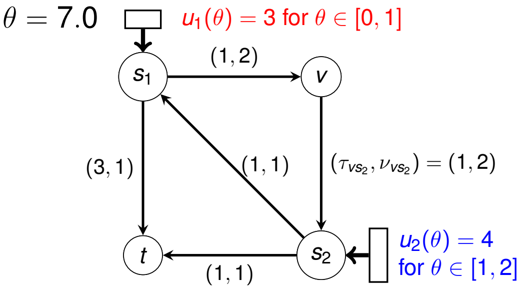

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Dynamic Nash – Single Sink

- *total travel time* of edge e at time θ: $c_e(\theta) = \tau_e + q_e(\theta)/\nu_e$
- exit time $T_e(\theta) = \theta + c_e(\theta)$.
- node labels $\ell_v(\theta)$: earliest arrival time at t from v at θ:

$$\ell_v(\theta) = \begin{cases} \theta, & \text{for } v = t \\ \min_{e=vw\in E}\{\ell_w(T_e(\theta))\}, & \text{for all } v \in V \setminus \{t\}. \end{cases}$$

Definition 6 (Active Edges)

An edge $e = vw \in E$ is active at time θ if

$$\ell_v(\theta) = \ell_w(T_e(\theta)).$$

$E(\theta) \subseteq E$ set of active edges

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Definition 7 (Dynamic Equilibrium)

For every $\theta \geq 0$: $f_e^+(\theta) > 0 \Rightarrow e \in E(\theta)$.

Related Works

- Nash Equilibrium
  - Koch and Skutella ('11)
  - Cominetti, Correa and Larre ('15)
  - Cominetti, Correa and Olver ('17)
  - Sering, Skutella (18)
  - Sering, Vargas-Koch, Olver ('19, '22)

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- Stochastic Equilibrium, Day-to-Day Learning, bounded rationality
  - Cantarella and Watling (2006+)
  - Yu, Han, Ochien (2020)
  - path/route based DTA !

Queue Predictions

So far:

Fixed time $\bar{\theta}$:

- full information model: $q_e(\theta)$ is known for all $\theta \geq \bar{\theta}$.
- instantaneous model: $q_e(\bar{\theta})$ is known for current $\bar{\theta}$.

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Prediction Functions (PF)

A PF is a mapping $\hat{q}_{i,e}(\theta; \bar{\theta}; f)$ with signature:

$$\hat{q}_{i,e} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times (\mathcal{R} \times \mathcal{R})^{I \times E} \to \mathbb{R}_{\geq 0}$$

where $\mathcal{R} := L_1^{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$.

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- θ current time
- $\bar{\theta}$ time of prediction
- f (historical) flow

Properties of Prediction Functions

Definition 8

A PF $\hat{q}_{i,e}$ is *p-continuous* for some $p > 1$ if the mapping

$$f^+ \mapsto \hat{q}_{i,e}(\,\cdot\,,\,\cdot\,,f)$$

is sequentially weak-strong continuous from $L^p([0,M])^{I \times E}$ to $C([0,M] \times D)$ for all $M > 0$ and compact intervals D.

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A PF $\hat{q}_{i,e}$ is *oblivious*, if for all $\bar{\theta} > 0$ it holds

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Definition 10

A PF $\hat{q}_{i,e}$ is *FIFO*, if $\theta \mapsto \theta + \frac{\hat{q}_{i,e}(\theta, \bar{\theta}, f)}{\nu_e}$ is monotone.

Prediction Equilibria

- edge $e = vw$ at time θ
- travel time predicted at time $\bar{\theta} \leq \theta$ is:

$$\hat{c}_{i,e}(\theta; \bar{\theta}; f) := \tau_e + \frac{\hat{q}_{i,e}(\theta; \bar{\theta}; f)}{\nu_e}$$

- predicted exit time

$$\hat{T}_{i,e}(\theta; \bar{\theta}; f) := \theta + \hat{c}_{i,e}(\theta; \bar{\theta}; f).$$

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label functions = earliest predicted arrival:

$$\hat{\ell}_{i,v}(\theta; \bar{\theta}; f) = \begin{cases} \theta & \text{if } v = t_i, \\ \min\limits_{vw \in \delta_v^+} \hat{\ell}_{i,w}(\hat{T}_{i,vw}(\theta; \bar{\theta}; f); \bar{\theta}; f) & \text{if } v \neq t_i. \end{cases} \qquad (3)$$

Existence of DPE

We say that an edge $e = vw$ is $\bar{\theta}$*-estimated active* at time θ, if

$$\hat{\ell}_{i,v}(\theta; \bar{\theta}; f) = \hat{\ell}_{i,w}\Big(\hat{T}_{i,e}(\theta; \bar{\theta}; f); \bar{\theta}; f\Big).$$

Definition 11

A tuple $(\hat{q}, f)$ of PF and dynamic flow is a *dynamic prediction equilibrium (DPE)*, if for all $e \in E$, $i \in I$ and $\theta \geq 0$:

$$f_{i,e}^{+}(\theta) > 0 \implies e \in \hat{E}_i(\theta; \theta; f),$$

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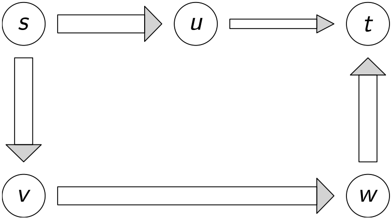
Theorem 12 (Graf, H., Kollias, Markl)

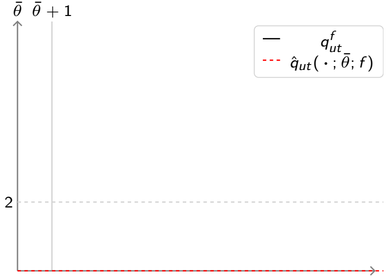
For p-continuous, oblivious, and FIFO PFs, there is a DPE.

Proof: ϵ-extension via solution of variational inequality in a reflexive Banach space.

An example DPE

- Linear predictor $\hat{q}^L_{i,e}(\theta, \bar{\theta}, f) := q^f_e(\theta) + \partial_- q^f_e(\theta) \cdot \left(\theta - \bar{\theta}\right)$





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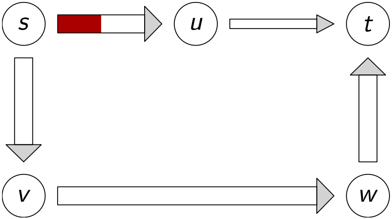

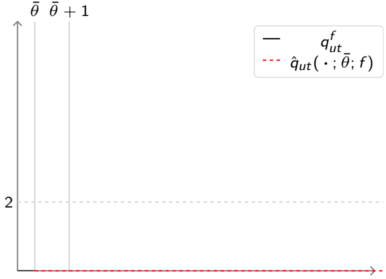

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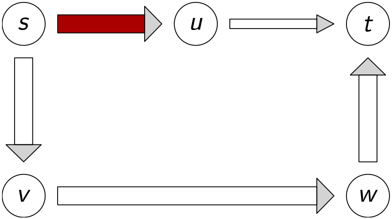

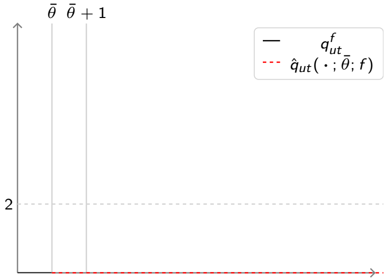

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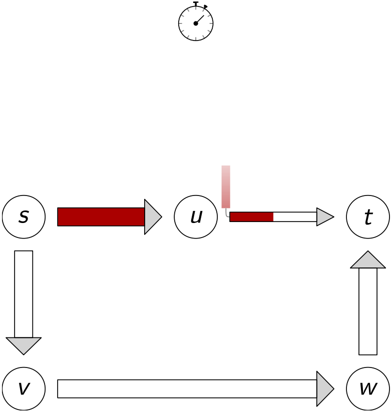

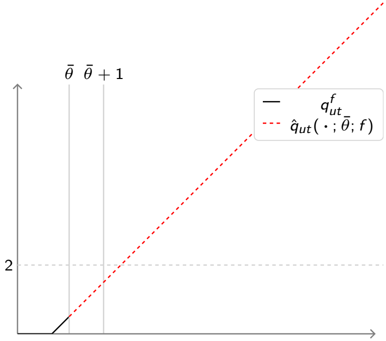

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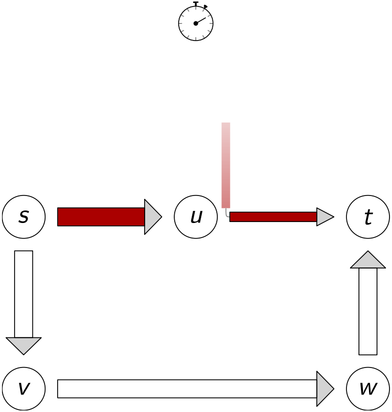

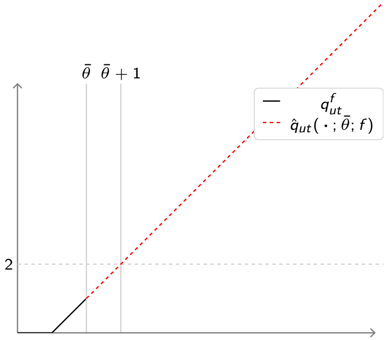

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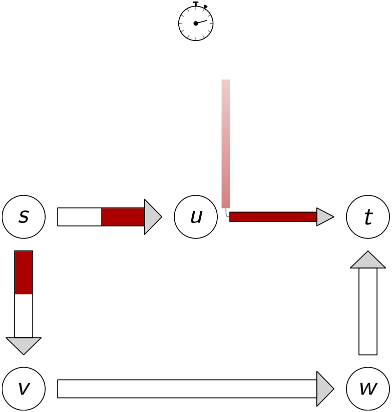

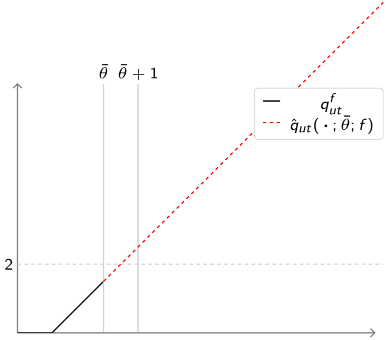

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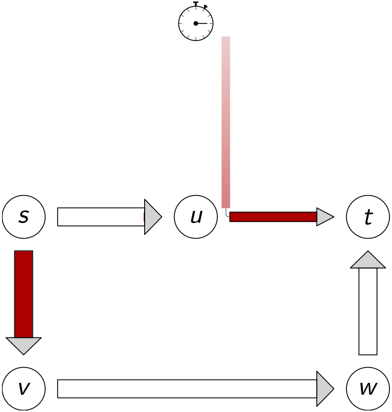

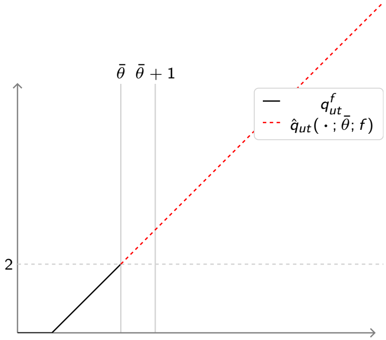

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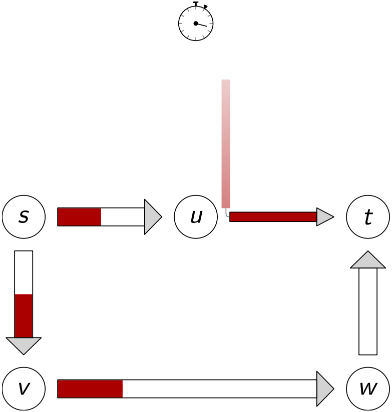

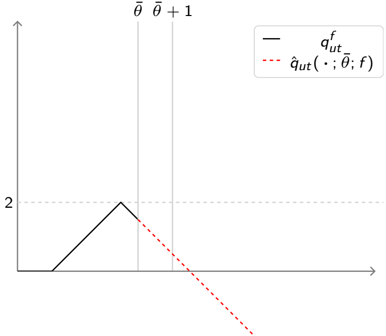

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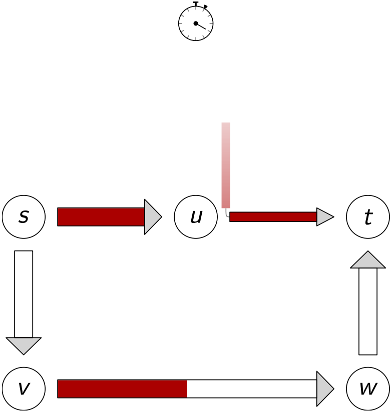

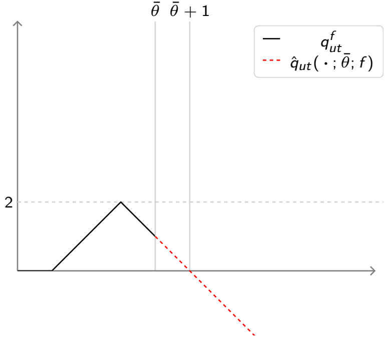

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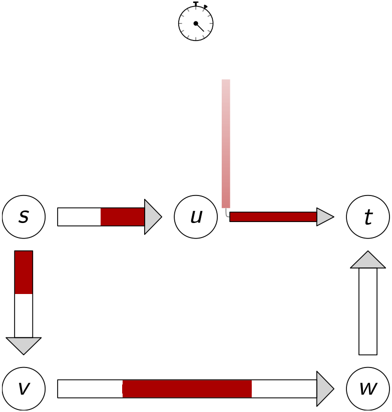

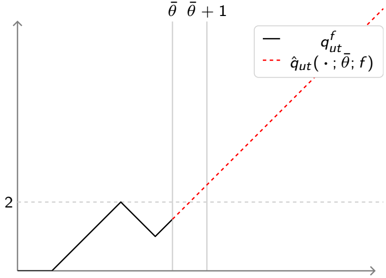

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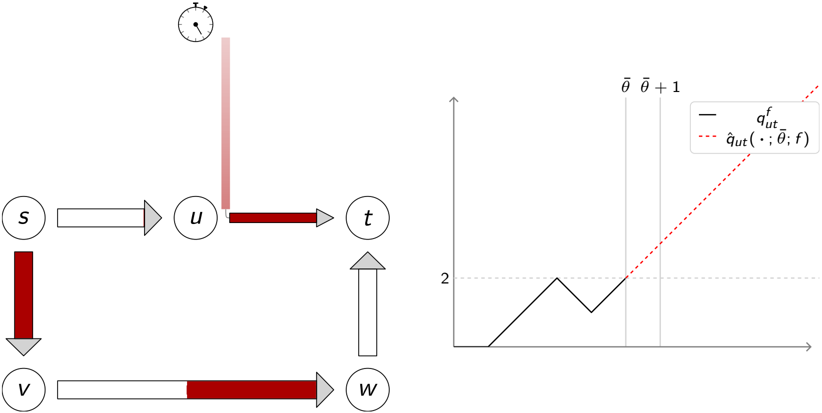

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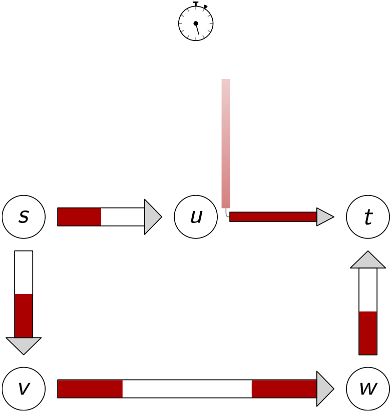

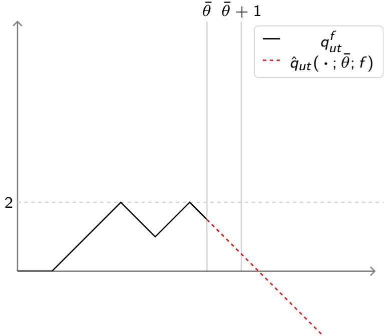

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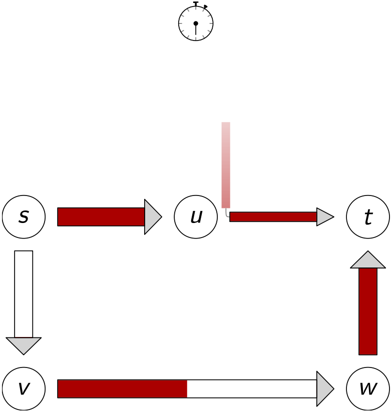

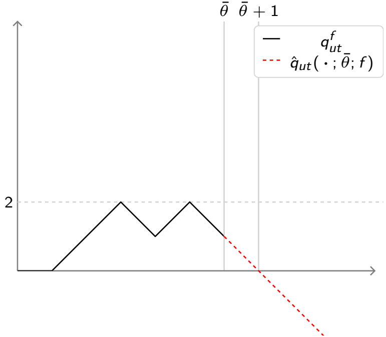

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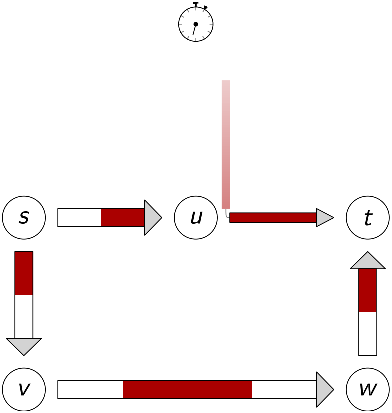

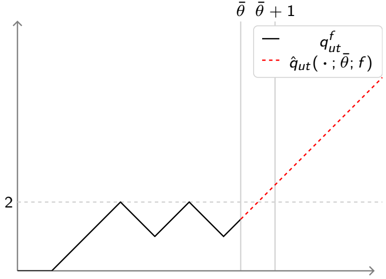

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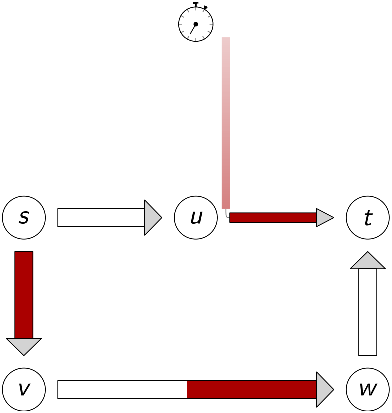

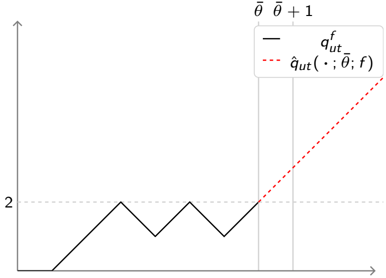

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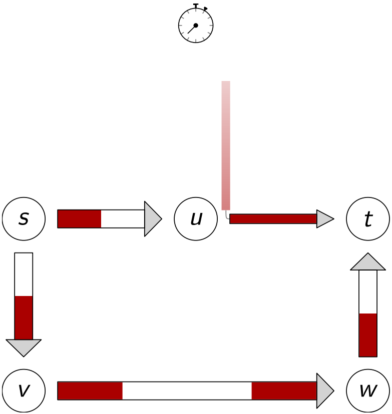

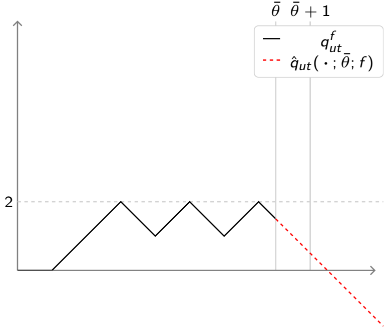

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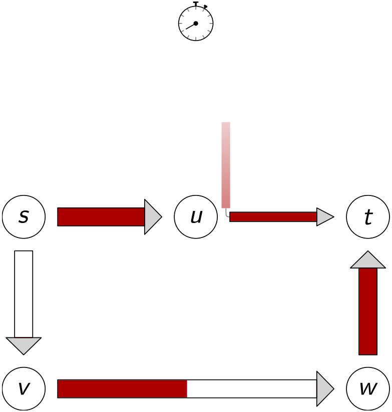

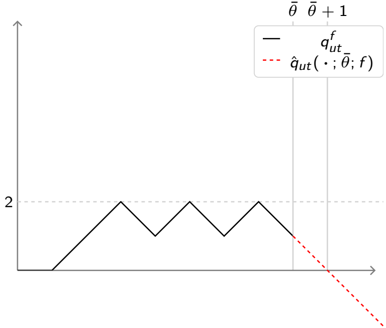

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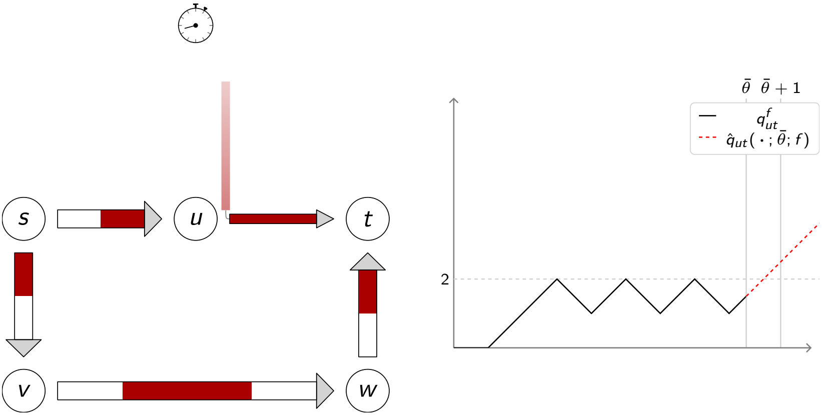

Example PFs

- Zero-predictor: $\hat{q}^{\mathsf{Z}}_{i,e}(\theta;\bar{\theta};f) = 0$.
- Constant predictor: $\hat{q}^{\mathsf{C}}_{i,e}(\theta;\bar{\theta};f) = q^f_e(\bar{\theta})$.
- Linear Predictor with horizon $H > 0$:

$$\hat{q}^{\mathsf{L}}_{i,e}(\theta;\bar{\theta};f) := \left(q^f_e(\bar{\theta}) + \partial_- q^f_e(\bar{\theta}) \cdot \min\{\theta - \bar{\theta}, H\}\right)^+,$$

- Regularized Linear Predictor:

$$\hat{q}^{\mathsf{RL}}_{i,e}(\theta;\bar{\theta};f) := \left(q^f_e(\bar{\theta}) + \frac{q^f_e(\bar{\theta}) - q^f_e(\bar{\theta} - \delta)}{\delta} \cdot \min\{\theta - \bar{\theta}, H\}\right)^+$$

- Machine-Learned Predictors

Machine-learned predictors

$$\Big(\theta, \bar{\theta}, f\Big)$$

$$\Big\downarrow \sigma \quad \textit{(extracts past samples of the queues and edge loads)}$$

$$\begin{pmatrix} q^f_{e'}(\bar{\theta} - j\delta), \\ L^f_{e'}(\bar{\theta} - j\delta) \end{pmatrix}_{\substack{e' \in N(e) \subseteq E \\ j \in \{0,\dots,k_p - 1\}}}$$

$$\Big\downarrow \phi_{i,e} \quad \textit{(applies a machine-learned transformation)}$$

$$\Big(\hat{q}^{\mathrm{ML,raw}}_{i,e}(\bar{\theta} + j\delta, \bar{\theta}, f)\Big)_{j \in \{1,\dots,k_f\}}$$

Machine-learned predictors

$$\left(\theta, \bar{\theta}, f\right)$$

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- The predicted points are linearly interpolated.

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$$\left(\hat{q}^{\mathrm{ML,raw}}_{i,e}(\bar{\theta} + j\delta, \bar{\theta}, f)\right)_{j \in \{1, \ldots, k_f\}}$$

- The predicted points are linearly interpolated.
- With appropriate post-processing, any continuous $\phi_{i,e}$ yields a p-continuous, oblivious, and FIFO PF.

Computational Study

Two machine-learned PFs:

- $\hat{q}_{i,e}^{\mathrm{LR}}$: Linear Regression
- $\hat{q}_{i,e}^{\mathrm{NN}}$: 4-layered Dense Neural Network

Input Features: $k_p = 20$ samples of edges at most 3 jumps away

Output: $k_f = 20$ samples of the predicted queue length

Training Data: Computed DPE using the constant predictor exclusively

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Evaluation:

- Compute approx. DPE with all predictors used side by side.
- Monitor average travel time T_i^{avg} of each predictor.
- Compare with optimal average travel time $T_{\mathrm{OPT}}^{\mathrm{avg}}$.

Computational Study: Results

Network	$\lvert E\rvert$	$\lvert V\rvert$	$\lvert I\rvert$
Synthetic	5	4	1
Nguyen	19	13	4
Sioux Falls	75	24	528
Anaheim	914	416	1.406

Table. Attributes of the considered networks.

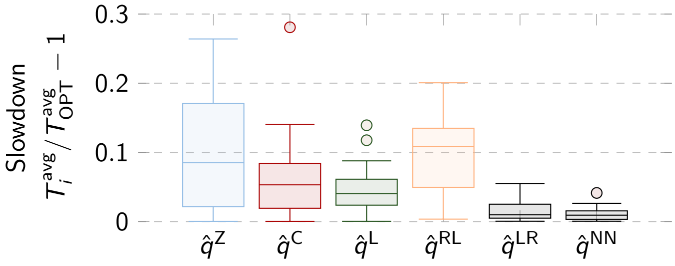

Figure. Slowdown in the Anaheim network

Summary

- IDE Flows

	Existence	Termination	Cycling
Single-Sink	Yes	Yes	Yes
Multi-Sink	Yes	No	Yes

- DE Flows

	Existence	Termination	Cycling
Single-Sink	Yes	Yes	No
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- DPE: Existence for p-continuous, oblivious, and FIFO PFs

Problem 4

- For which PFs finite Termination ?
- Good ML-models ?

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Thank You!