Contagion in Financial Networks

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Introduction

What the 2008 crisis taught us...

- High interconnectedness of modern financial system;
- default risk of a bank depends on the whole set of connections (network);
- the network topology can trigger default cascade and shock's amplification effects.

Main challenges

- Defining a network model that accounts for propagation effects;
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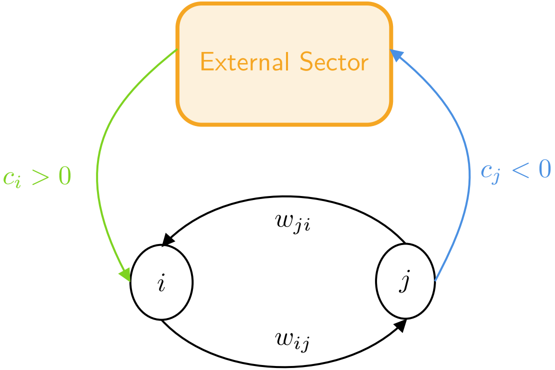

- w_{ij} inter-bank liability;
- $c_i > 0$ positive money inflow;
- $c_j < 0$ outside debt.

Everything is fine

In normal conditions, every bank i can meet its total liability: $w_i = \sum_j w_{ij}$.

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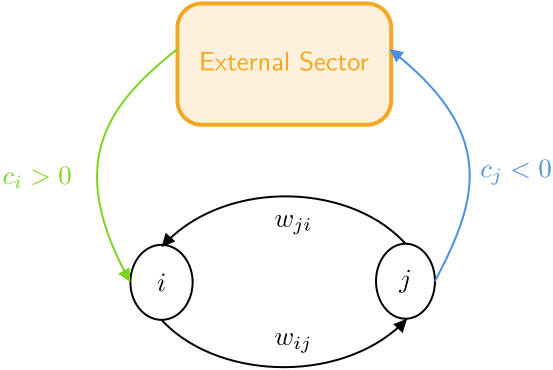

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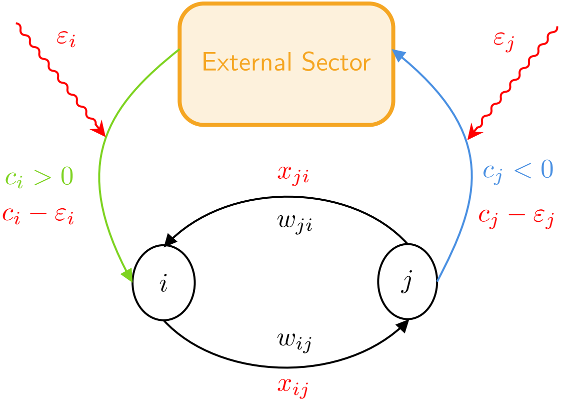

- Shocks ε hit the network by reducing c;
- Nodes may default and not be able to pay their liabilities (direct effect);
- Shocks propagate across the network because of reduced payments (indirect effect).

Clearing Vectors

x is a set of consistent payments after the shock:

$$x = \mathcal{S}_0^w(P^\top x + c - \varepsilon)$$

where $(P)_{ij} = \frac{w_{ij}}{w_i}$ and $\mathcal{S}_0^w$ is a saturation:

- Notice that any solution is such that $x \in \mathcal{L}_0^w := \{x \in \mathbb{R}^n \ : \ 0 \leq x \leq w\}$

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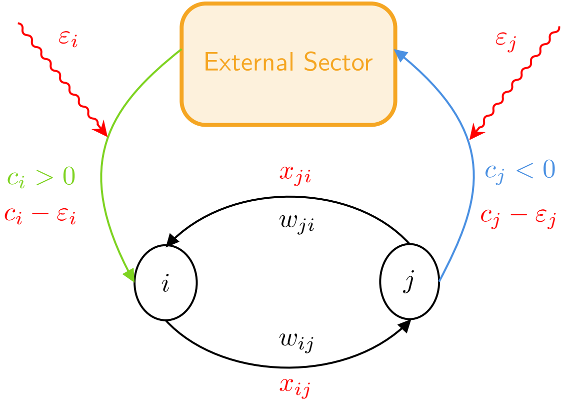

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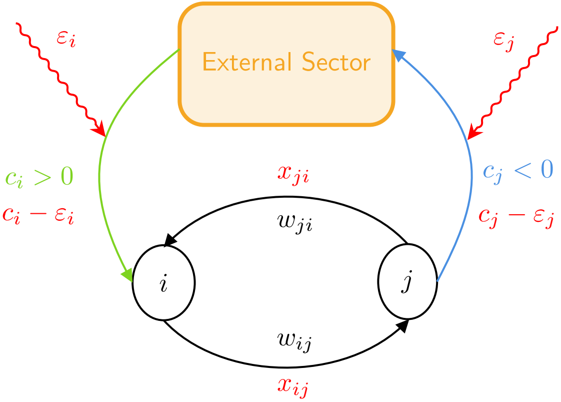

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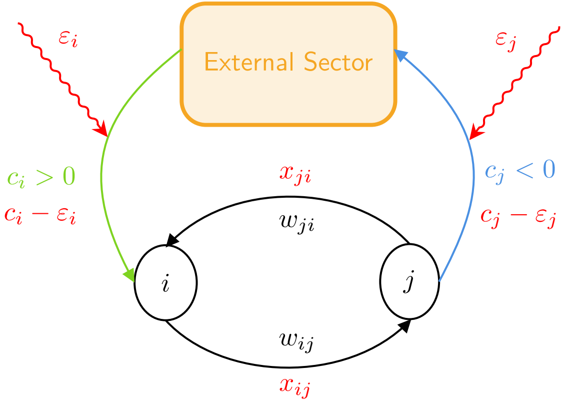

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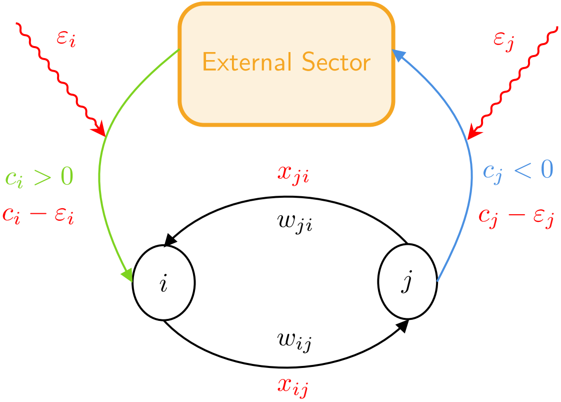

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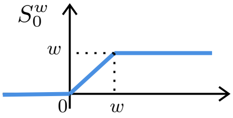

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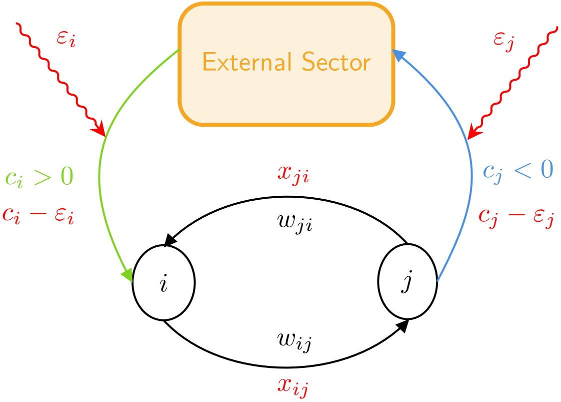

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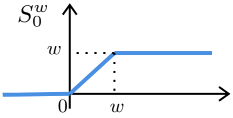

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The saturated equilibrium model[1]

We study saturated equilibrium models in networks. Precisely, we consider the following fixed point equation

$$x_i = \min\left\{\max\left\{\sum_{j=1}^{n} x_j P_{ji} + c_i, 0\right\}, w_i\right\}, \quad i = 1, \dots, n$$

or, more compactly,

$$x = S_0^w\left(P^\top x + c\right)$$

where:

- $\left(S_0^w(x)\right)_i = \min\left\{\max\left\{x_i, 0\right\}, w_i\right\}, \quad i = 1, \dots, n,;$
- $P \in \mathbb{R}_+^{n\times n}$ is a non-negative square matrix and $w \in \mathbb{R}_+^n$ that jointly describe the network;
- the solutions $x \in \mathcal{X}$ are called equilibria of the network (P, w) with exogenous flow c;
- $x \in \mathcal{L}_0^w = \{x \in \mathbb{R}^n : 0 \leq x \leq w\}$

[1] L. Massai, G. Como, and F. Fagnani. "Equilibria and Systemic Risk in Saturated Networks". In: *Mathematics of Operations Research* (2021). URL: https://doi.org/10.1287/moor.2021.1188.

Network games with monotone linear saturated best responses

Consider a set of players $\mathcal{V} = \{1, \dots, n\}$ playing an action $x_i \in [0, w_i]$ and with quadratic utility

$$u_i(x) = u_i\left(x_i, x_{-i}\right) = c_i x_i - \frac{x_i^2}{2} + x_i \sum_j P_{ji} x_j$$

- $P_{ij} \geq 0$ strength of interaction: games of pure strategic complements.

Quadratic utility $\implies$ best response of a player i is always unique and given by

$$B_i\left(x_{-i}\right) = \min\left\{\max\left\{\sum_{j=1}^{n} x_j P_{ji} + c_i, 0\right\}, w_i\right\}.$$

- Nash equilibria are exactly such that $x = S_0^w(P^\top x + c)$;
- more in general, our analysis applies to $u_i(x) = \varphi_i\left(x_i - c_i + \sum_{j \neq i} P_{ji} x_j\right)$ for a continuous $\varphi_i : \mathbb{R} \mapsto \mathbb{R}$ that is increasing on $(-\infty, 0]$ and decreasing in $[0, +\infty)$;
- these are *supermodular games* (increase of one player's action encourages the others to do so as well).

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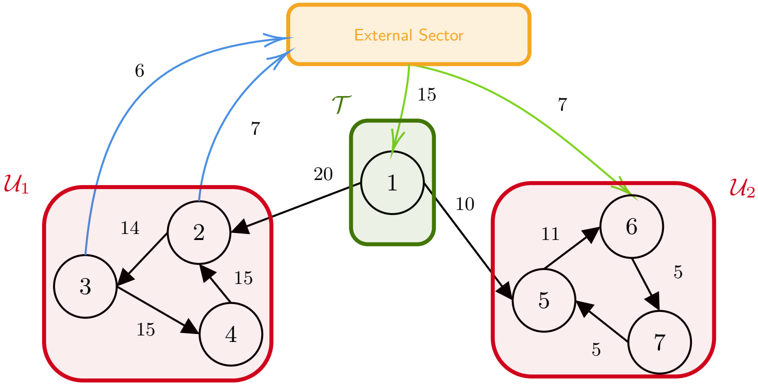
Iterative algorithms and contagion spreading

Example

$$\begin{cases} x(t+1) & = S_0^w(P^\top x(t) + c) \\ x(0) & = w \end{cases}$$



Iterative algorithms and contagion spreading

Example

$t = 0$

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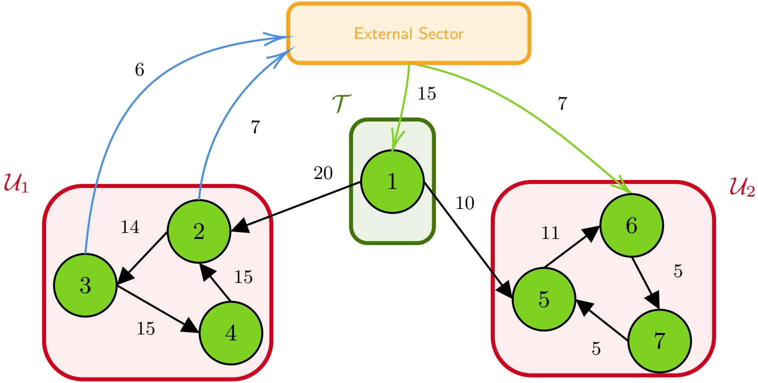

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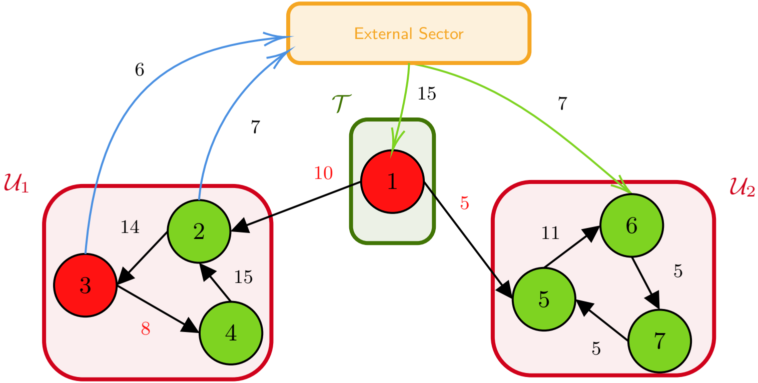

Iterative algorithms and contagion spreading

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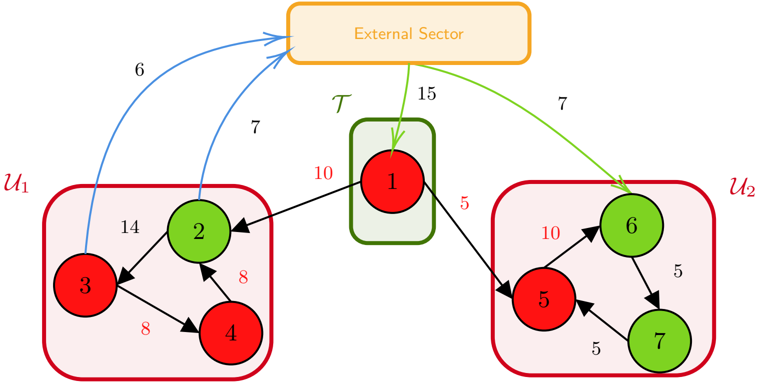
$t = 2$

$$\begin{cases} x(t+1) &= S_0^w(P^\top x(t) + c) \\ x(0) &= w \end{cases}$$



Iterative algorithms and contagion spreading

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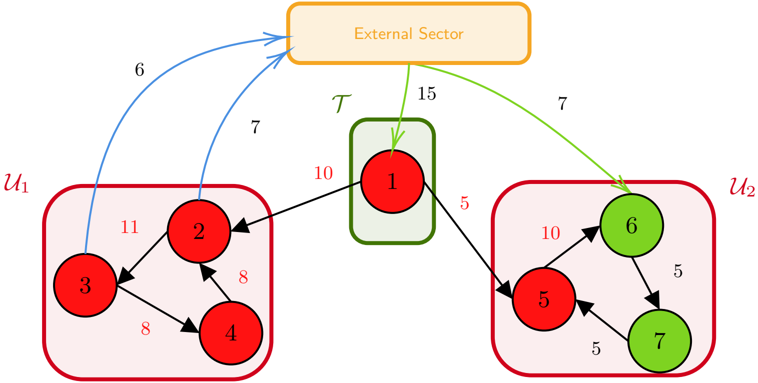

Iterative algorithms and contagion spreading

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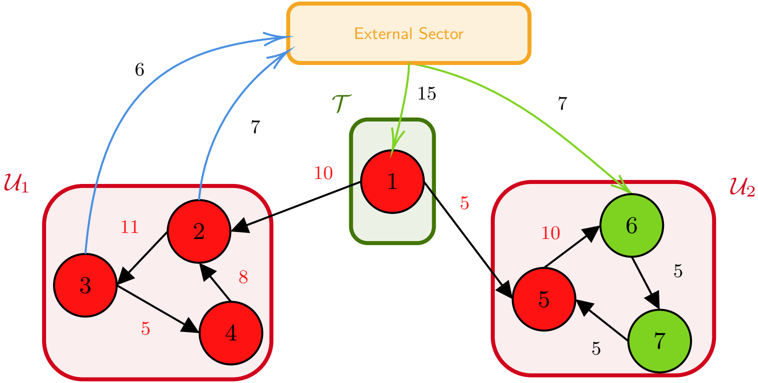

Iterative algorithms and contagion spreading

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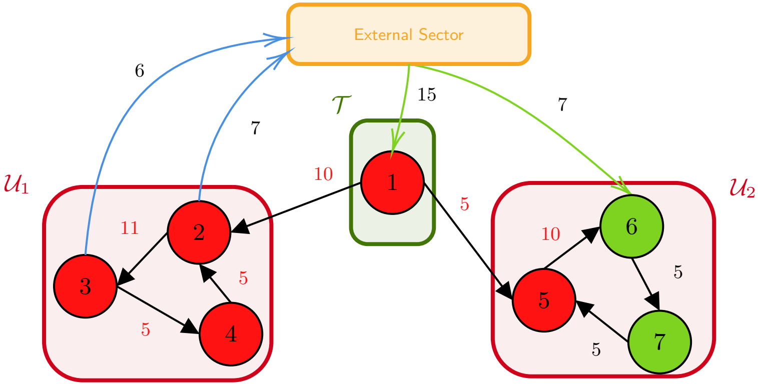

Iterative algorithms and contagion spreading

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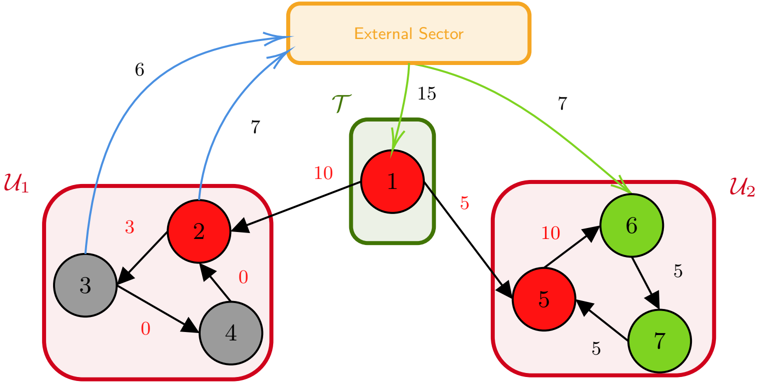

Existence and uniqueness of equilibria

- Existence of equilibria follows from Brower fixed point Theorem.

In general however the equilibrium will not be unique:

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Consider the network described by $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

It is immediate to check that any $x = \begin{bmatrix} t \\ t \end{bmatrix}$, $t \in [0, 1]$ is an equilibrium.

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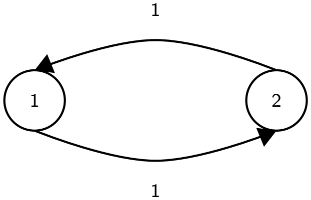

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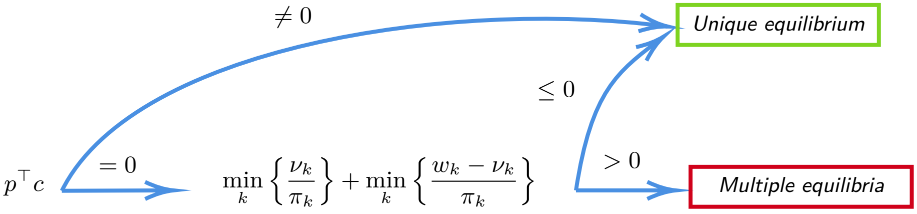
The irreducible case

Theorem (Uniqueness for the irreducible case (Massai, Como, Fagnani, 2021))

Let (P, w) be a network such that P is irreducible and $\rho(P) = 1$. Let π and p be, respectively, left and right dominant eigenvectors of P. Let ν be any solution of $\nu = P^\top \nu + c$. Then it holds:



- In case we have multiple equilibria, the set of equilibria $\mathcal{X}$ is:

$$\mathcal{X} = \left\{ x = \nu + \alpha\pi \;:\; -\min_k \left\{ \frac{\nu_k}{\pi_k} \right\} \leq \alpha \leq \min_k \left\{ \frac{w_k - \nu_k}{\pi_k} \right\} \right\}$$

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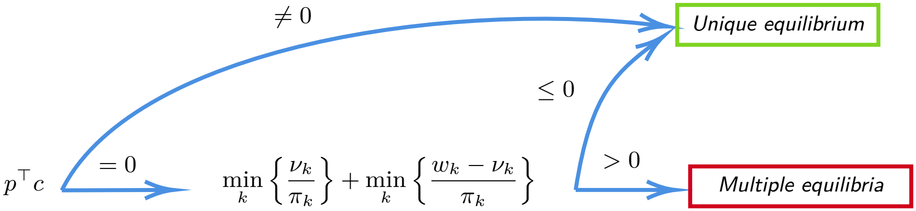

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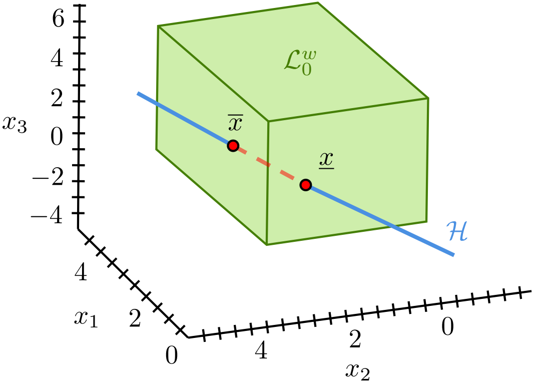
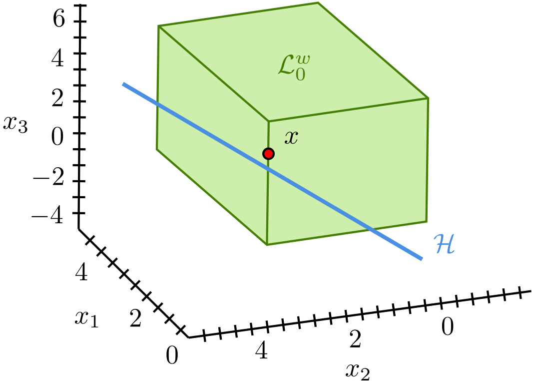
A geometrical interpretation

When $p^\top c = 0$, we have multiple equilibria when the line $\mathcal{H} = \{x \in \mathbb{R}^n \ : \ x = \nu + \alpha\pi, \alpha \in \mathbb{R}\}$ intersects non trivially the lattice $\mathcal{L}_0^w$.



(a) Multiple equilibria (the red dots and segment).

$$\min_k \left\{\frac{\nu_k}{\pi_k}\right\} + \min_k \left\{\frac{w_k - \nu_k}{\pi_k}\right\} > 0$$



(b) Unique equilibrium (the red dot).

$$\min_k \left\{\frac{\nu_k}{\pi_k}\right\} + \min_k \left\{\frac{w_k - \nu_k}{\pi_k}\right\} \leq 0$$

The dependence of equilibria on the flow

- Uniqueness ultimately depends on exogenous flow c.
- There exists a set $\mathcal{M}$ of critical vectors c^* such that we have multiple solutions, namely: $\mathcal{U} = \{c \in \mathbb{R}^n : |\mathcal{X}(c)| = 1\}, \qquad \mathcal{M} = \mathbb{R}^n \setminus \mathcal{U}$

Theorem (Continuity of network equilibria (Massai, Como, Fagnani, 2021))

For a network (P, w) such that $\rho(P) \leq 1$, let m be number of basic classes of P. Then,

(i) *the non-uniqueness set $\mathcal{M}$ has Lebesgue measure 0 and is contained in the closed set consisting of the union of at most m graphs of scalar continuous functions;*

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Jump discontinuity $\implies$ slight change of c may lead to a catastrophic aggregated loss and to sudden defaults of several nodes.

Loss function

- Take an asset c° such that the system is healthy $(x(c^\circ) = w)$ and another $c < c^\circ$ after a shock;
- Net worth before the shock: $v^\circ = P^\top w + c^\circ - w$;
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$$l\left(c^\circ, c\right) := \mathbb{1}^\top \left(v^\circ - v\right) = \mathbb{1}^\top \left(P^\top w + c^\circ - w - \left(P^\top x(c) + c - w\right)\right) = \underbrace{\mathbb{1}^\top \left(c^\circ - c\right)}_{\text{direct loss}} + \underbrace{\mathbb{1}^\top \left(w - x(c)\right)}_{\text{shortfall term}}$$

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$$\Delta l\left(c^*\right) := \limsup_{\substack{c \in U \\ c \to c^*}} l\left(c^\circ, c\right) - \liminf_{\substack{c \in \mathcal{U} \\ c \to c^*}} l\left(c^\circ, c\right) = \left\|\bar{x}\left(c^*\right) - \underline{x}\left(c^*\right)\right\|_1 .$$

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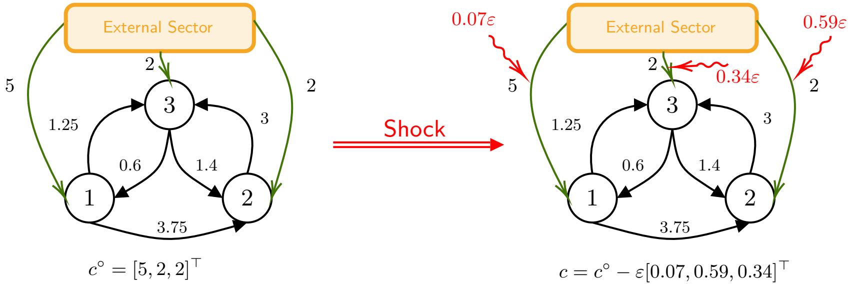
Example

- Consider a network with $P = \begin{bmatrix} 0 & 0.75 & 0.25 \\ 0 & 0 & 1 \\ 0.3 & 0.7 & 0 \end{bmatrix}$ and $w = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$.
- Consider an initial vector $c^\circ = [5, 2, 2]^\top$ that we perturb with a shock ε such that:

$$c = c^\circ - \varepsilon v, \qquad v = \begin{bmatrix} 0.07 \\ 0.59 \\ 0.34 \end{bmatrix}, \quad \varepsilon \in [0, 14]$$



Systemic risk in financial networks

Example

- We expect a jump discontinuity when $\mathbb{1}^\top c = 0 \implies \varepsilon = 9 \implies c^* = [4.4, -3.3, -1.1]^\top$.

- The size of the jump is $\Delta l\,(c^*) = \min_i \left\{\frac{\nu_i}{\pi_i}\right\} + \min_i \left\{\frac{w_i - \nu_i}{\pi_i}\right\} \approx 4.44$
- At c^* the network suffers a dramatic crisis as node 1 and 3 suddenly default.
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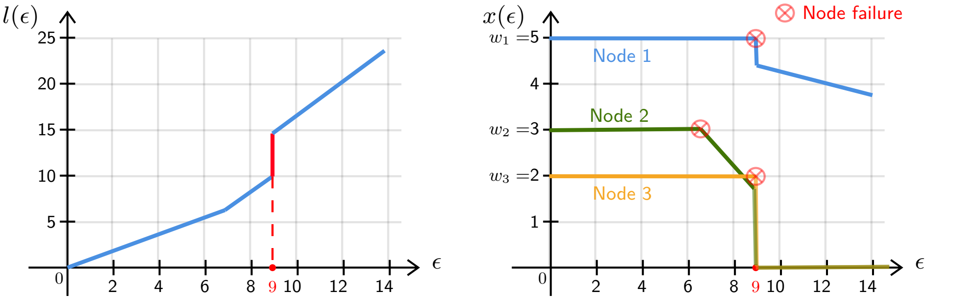

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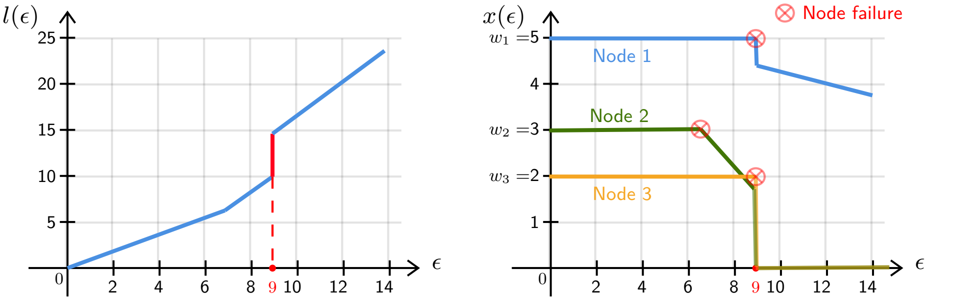

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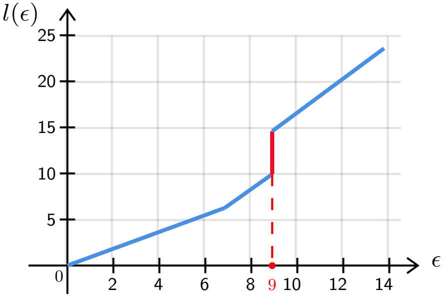

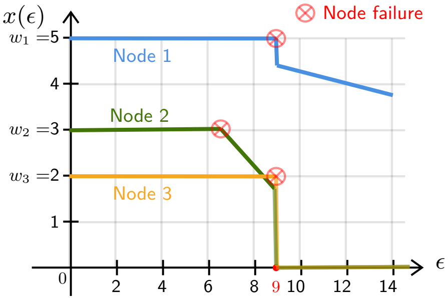

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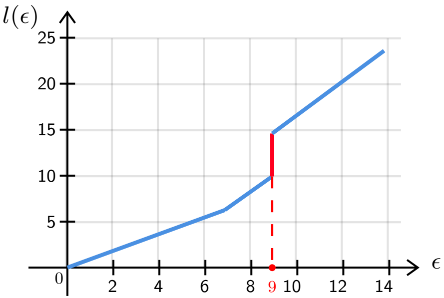

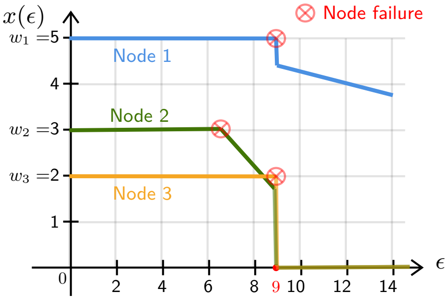

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Sensitivity of Nash equilibria in constrained quadratic network games

- $P^{(\delta)} = \delta G$ where G is some fixed matrix and $\delta > 0$ describes the strength of interactions among players.
- If we put $\delta^* = \rho(G)^{-1}$, we have that $\rho(\delta G) < 1$ for $\delta < \delta^*$.
- The sensitivity of the unique Nash equilibrium to the variation of c may grow unbounded when δ approaches δ^* if the limit network admits multiple equilibria.

Proposition (Sensitivity of Nash equilibria (Massai, Como, Fagnani, 2021))

Let $P^{(\delta)} = \delta G$ with G irreducible, $\delta \in (0, \delta^*]$ and let $\bar{x}^{(\delta)}(c)$ and $\underline{x}^{(\delta)}(c)$ to be the min. and max. equilibria of $\left(P^{(\delta)}, w\right)$ with $c \in \mathbb{R}^n$. Write $x^{(\delta)}$ for the equilibrium when it is unique. Let c^* be such that the $\left(P^{\delta^*}, w\right)$ has multiple network equilibria. Then,

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Ongoing research

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Thank you for your attention!

A Fundamental Partition

To every solution $x \in \mathcal{X}$, we attach a node partition: a node $i \in \mathcal{V}$ is called a

The Fundamental Partition

- $i \in \mathcal{V}_x^+$ *surplus node* if $c_i + \sum_{k \neq i} P_{ki} x_k > w_i \implies x_i = w_i$;
- $i \in \mathcal{V}_x^0$ *exposed node* if $0 \leq c_i + \sum_{k \neq i} P_{ki} x_k \leq w_i \implies x_i = c_i + \sum_{k \neq i} P_{ki} x_k$;
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Theorem (Invariance of the Fundamental Partition)

The partition $\mathcal{V}_x^-, \mathcal{V}_x^+, \mathcal{V}_x^0$ is invariant over all solutions $x \in \mathcal{X}$.

The unique partition of nodes can be denoted with the triple $\mathcal{V}^+, \mathcal{V}^0, \mathcal{V}^-$. For every x we have:

$$x = \begin{bmatrix} x_{\mathcal{V}^+} \\ x_{\mathcal{V}^0} \\ x_{\mathcal{V}^-} \end{bmatrix} = \begin{bmatrix} w_{\mathcal{V}^+} \\ x_{\mathcal{V}^0} \\ 0 \end{bmatrix}$$

- What remains to be studied is the structure of the solutions on $\mathcal{V}^0$.
- Notice that $x_{\mathcal{V}_0} = \mathcal{S}_0^w(P'_{\mathcal{V}_0} x_{\mathcal{V}_0} + c_{\mathcal{V}_0}) = P'_{\mathcal{V}_0} x_{\mathcal{V}_0} + c_{\mathcal{V}_0}$; a linear equation!

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The Out-connected Case

Theorem (Uniqueness for out-connected graphs)

Let P be an out-connected matrix, then the clearing vector is unique.

Proof.

By the invariance of the partition, we just need to check uniqueness for nodes in $\mathcal{V}_0$. We have $x_{\mathcal{V}_0} = P'_{\mathcal{V}_0} x_{\mathcal{V}_0} + c_{\mathcal{V}_0} \implies x_{\mathcal{V}_0} = \left(I - P'_{\mathcal{V}_0}\right)^{-1} c_{\mathcal{V}_0}$ and hence the solution is unique. Notice that $\left(I - P'_{\mathcal{V}_0}\right)$ is invertible since $P_{\mathcal{V}_0}$ is an out-connected matrix. □

- We can partition any graph in a transient part $\mathcal{T}$ and trapping sets $\mathcal{U}$. I.e. $\mathcal{V} = \mathcal{T} \cup (\cup_k \mathcal{U}_k)$;
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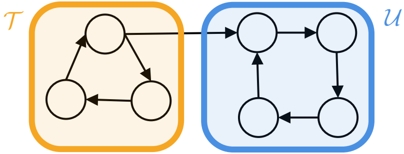

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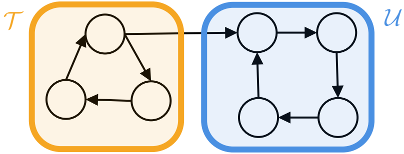

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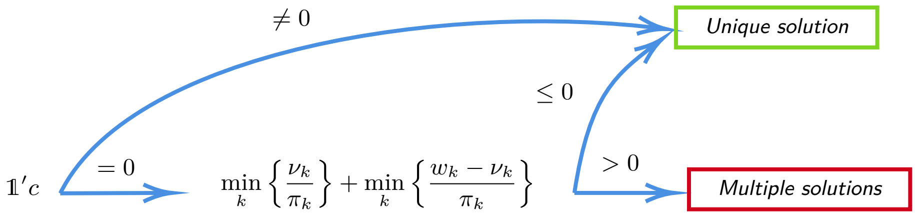
The Stochastic Irreducible Case

Theorem (Uniqueness for the stochastic irreducible case)

Let P be an irreducible stochastic matrix; let π *be its unique invariant probability measure and* $\nu = \frac{1}{2}\sum_{k\geq 0}\left(\frac{I+P'}{2}\right)^k c$. *Then it holds:*



- In case we have multiple solutions, we have that:

$$\mathcal{X} = \left\{ x = \nu + \alpha\pi \ : \ -\min_k \left\{\frac{\nu_k}{\pi_k}\right\} \leq \alpha \leq \min_k \left\{\frac{w_k - \nu_k}{\pi_k}\right\} \right\}$$

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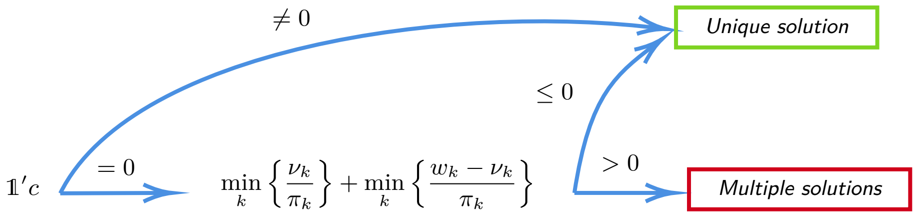

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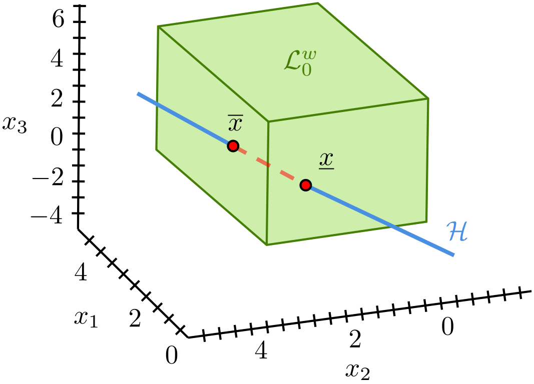
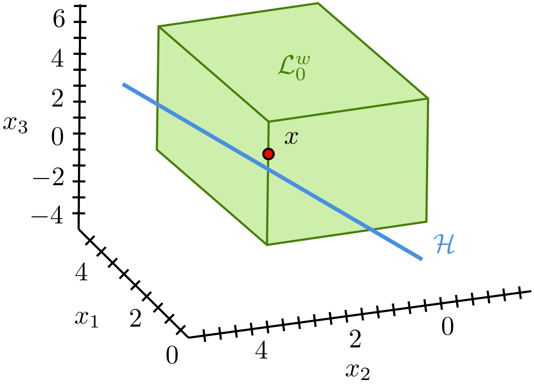
A geometrical Interpretation

When $\mathbb{1}'c = 0$, we have multiple solutions when the line $\mathcal{H} = \{x \in \mathbb{R}^n \,:\, x = \nu + \alpha\pi\}$ intersects non trivially the lattice $\mathcal{L}_0^w$. This corresponds to $\min_k \left\{\frac{\nu_k}{\pi_k}\right\} + \min_k \left\{\frac{w_k - \nu_k}{\pi_k}\right\} > 0$



(m) Multiple solutions (the red dots and the red segment).

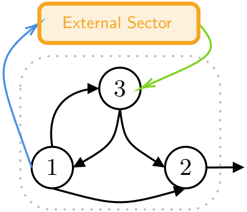


(n) Unique solution (the red dot).

The General Case

The Out-Connected Case

Unique solution.



The Stochastic-Irreducible Case

Uniqueness depends on c, i.e. on what is coming from and going to the external environment.

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- $x_{\mathcal{T}}$ is unique;
- For every trapping set $\mathcal{U}$, we use the Theorem;
- To do so, we also need to consider the input coming from $\mathcal{T}$: $h_{\mathcal{U}} := c_{\mathcal{U}} + P_{\mathcal{U}\mathcal{T}} x_{\mathcal{T}}$

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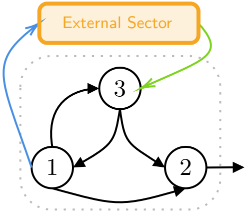

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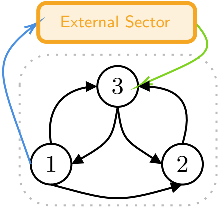

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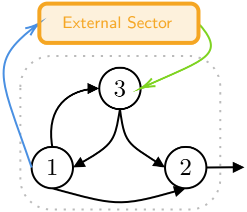

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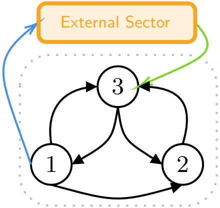

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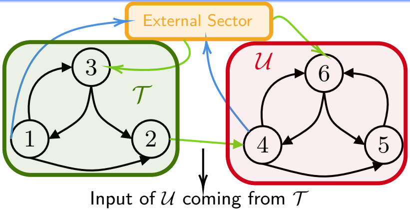

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The Dependence of Clearing Vectors on the Shock

Dependence of x on c

- The uniqueness ultimately depends on the input \ output vector c.
- There exists a set of critical vectors c^* such that we have multiple solutions, namely:

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What happens to the solutions when c approaches a critical $c^* \in \mathcal{M}$?

Let $\mathcal{A} = \mathbb{R}^n \backslash \mathcal{M}$ be the set where the solution is unique. Then:

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Critical Transitions and Jumps

Jump discontinuity as a financial breakdown

A jump discontinuity means that even a slight change in the asset/shock value c may lead to a catastrophic aggregated loss and to sudden defaults of several nodes.

Loss function

- Consider shock ε that lowers the value of the external asset from c to $c - \varepsilon$;
- Loss function is: $l = \mathbb{1}'(\varepsilon + w - x)$

Jump size of the loss function at $c^* \in \mathcal{M}$

$$\Delta l(c^*) = \liminf_{\substack{c \in \mathcal{A} \\ c \to c^*}} l(c) - \limsup_{\substack{c \in \mathcal{A} \\ c \to c^*}} l(c) = \mathbb{1}' \left(\bar{x}(c^*) - \underline{x}(c^*) \right) = \min_k \left\{ \frac{\nu_k}{\pi_k} \right\} + \min_k \left\{ \frac{w_k - \nu_k}{\pi_k} \right\}$$

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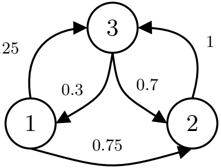
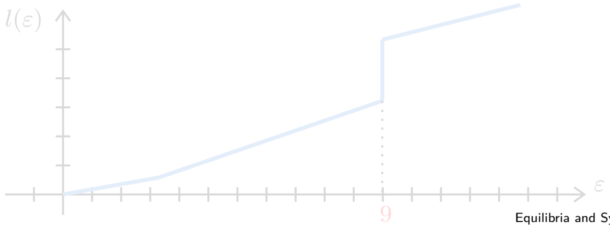
Critical Transitions and Jumps

Example

Consider the network below with $P = \begin{bmatrix} 0 & 0.75 & 0.25 \\ 0 & 0 & 1 \\ 0.3 & 0.7 & 0 \end{bmatrix}$ and $w = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$.



Consider an initial asset $c = [5, 2, 2]'$ and a total shock magnitude $\varepsilon \in [0, 12]$ that hits all nodes uniformly, i.e. $c(\varepsilon) = c - \varepsilon[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]'$. We expect a jump discontinuity when $\mathbb{1}'c(\varepsilon) = 0 \implies \varepsilon = 9$.



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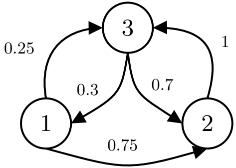
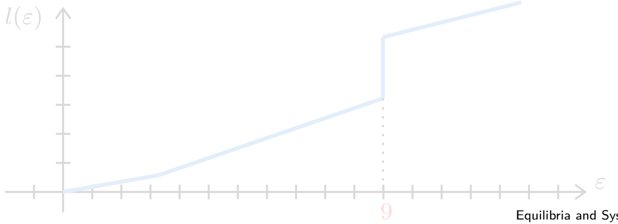


Consider an initial asset $c = [5, 2, 2]'$ and a total shock magnitude $\varepsilon \in [0, 12]$ that hits all nodes uniformly, i.e. $c(\varepsilon) = c - \varepsilon[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]'$. We expect a jump discontinuity when $\mathbb{1}'c(\varepsilon) = 0 \implies \varepsilon = 9$.



Critical Transitions and Jumps

Example

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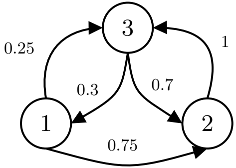

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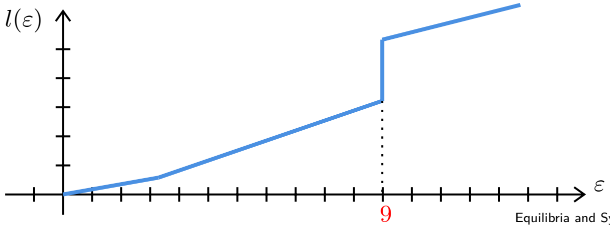

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- Sufficient and necessary condition for Uniqueness of clearing vectors;
- Systemic risk measures and existence of critical shocks;
- Structure of solutions with respect to the topological property of the network.

Ongoing Research

- Optimal policies for risk reduction;
- Analytical results on particular topologies and random graphs;
- Continuous Model.
- Model extensions (fire sales, bankruptcy costs, cross holdings, etc...);

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Thank you!