Best-response dynamics in two-person random games with correlated payoffs

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- The mixed extension of every finite game admits a Nash equilibrium.
- Some properties hold generically for finite games.
- Generically, finite games have a finite and odd number of Nash equilibria.
- Some properties are neither generically true nor generically false.

pure Nash equilibria (PNE)

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- Pure equilibria have a stronger epistemic foundation than mixed equilibria.
- Neither presence nor absence of pure equilibria is a generic property.
- How serious is the problem of lack of pure equilibria?
- One way to address the issue is to consider games with random payoffs.

Random games

- If we fix the set of players $[n]$ and the strategy set $[K]_i$ for each player i, a normal-form game is a point in $\mathbb{R}^H$ with

$$H = |[n]| \prod_{i \in [n]} |[K]_i|.$$

- A random game can be seen as a random vector with values in $\mathbb{R}^H$.
- This idea has been around, in one form or another for at least 65 years (Goldman(1957)).

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- Most of the literature fixes the set of players and the set of strategies for each player and assumes that the payoffs are i.i.d. from a continuous distribution.
- Exact results are cumbersome.
- Asymptotic results are easier to describe.

Asymptotic results

- Arratia, Goldstein, Gordon (1989) proved that, if the payoffs are i.i.d. from a continuous distribution, when the number of players is large, the random number of PNE has a distribution that is approximately Poisson(1).

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- Powers (1990) proved a similar result when the payoffs are i.i.d. from a continuous distribution and the number of strategies for at least two players is large.
- In both cases the technique that was used to prove the result was the Chen-Stein method.

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- Not much.
- Rinott and Scarsini (2000) proved that if payoff profiles are i.i.d. but dependence is allowed within the same payoff profile, then asymptotically in either the number of players or the number of strategies:
  - When there is negative dependence, the number of pure Nash equilibrium goes to zero,
  - When there is independence, the number of pure Nash equilibria is Poisson(1),
  - When there is positive dependence, the number of pure Nash equilibria diverges and a CLT holds.

Robustness, continued

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- Here we test robustness in different directions.

Robustness, continued

- Amiet et al. (2021) considered games with N players and 2 strategies for each player where the random payoffs are i.i.d. but their distribution may have atoms.
- They proved that the behavior of the number of pure Nash equilibria depends on one single parameter α: the probability of ties in the payoffs.
- Whenever $\alpha > 0$, the number of pure Nash equilibria diverges.

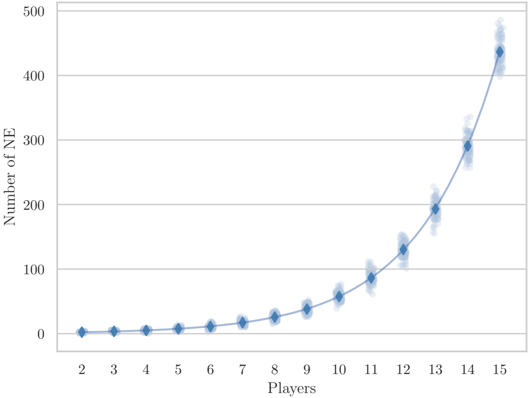

Figure 1: Number of PNE for $2 \leq N \leq 15$, $\alpha = 0.5$, with 100 trials per N. Diamond markers represent average number per value of N, and the curve $(1.5)^N$ is included for comparison.

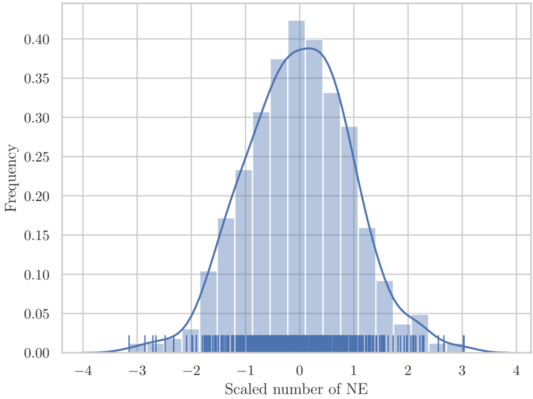

Figure 2: CLT result for $N = 15$, $\alpha = 0.9$, with 500 trials.

best-response dynamics (BRD)

- Amiet et al. (2021) studied also the behavior of best-response dynamics in this class of random games.
- They showed interesting phase-transition phenomena.
- They used percolation techniques to prove their results.

The model

- We consider two-person normal-form games.
- For $i \in \{A, B\}$, player i's action set is $[K^i]$,
- $U^i\colon [K^A] \times [K^B] \to \mathbb{R}$ is player i's payoff function.
- The game is defined by the payoff bimatrix

$$\boldsymbol{U} := (\boldsymbol{U}^A, \boldsymbol{U}^B).$$

- For $i \in \{A, B\}$,

$$\boldsymbol{U}^i := (U^i(a, b))_{a \in [K^{\mathbf{A}}], b \in [K^{\mathbf{B}}]}.$$

pure Nash equilibria

- A pure Nash equilibrium of the game is a pair (a^*, b^*) of actions such that, for all $a \in [K^\mathsf{A}], b \in [K^\mathsf{B}]$

$$U^\mathsf{A}(a^*, b^*) \geq U^\mathsf{A}(a, b^*) \quad \text{and} \quad U^\mathsf{B}(a^*, b^*) \geq U^\mathsf{B}(a^*, b).$$

Potential games

- A game is a potential game if there exists a potential function $\Psi\colon [K^{\mathsf{A}}] \times [K^{\mathsf{B}}] \to \mathbb{R}$ such that for all $a, a' \in [K^{\mathsf{A}}]$, for all $b, b' \in [K^{\mathsf{B}}]$,

$$U^{\mathsf{A}}(a,b) - U^{\mathsf{A}}(a',b) = \Psi(a,b) - \Psi(a',b),$$
$$U^{\mathsf{B}}(a,b) - U^{\mathsf{B}}(a,b') = \Psi(a,b) - \Psi(a,b').$$

- Potential games admit pure Nash equilibria.
- Every potential game is strategically equivalent to a common interest game, for instance to the game where $U^{\mathsf{A}} = U^{\mathsf{B}} = \Psi$.

Best-response dynamics

- (a_0, b_0) is a starting strategy profile.
- For each $t \geq 0$ $\mathsf{BRD}(t)$ is a process on $[K^{\mathsf{A}}] \times [K^{\mathsf{B}}]$ such that

  $$\mathsf{BRD}(0) = (a_0, b_0)$$

  and, if $\mathsf{BRD}(t) = (a', b')$, then, for t even,

  $$\mathsf{BRD}(t+1) = (a'', b'),$$

  where $a' \neq a'' \in \arg\max_{a \in [K^{\mathsf{A}}]} U^{\mathsf{A}}(a, b')$, if such an action a'' exists, otherwise

  $$\mathsf{BRD}(t+1) = \mathsf{BRD}(t);$$

  for t odd,

  $$\mathsf{BRD}(t+1) = (a', b''),$$

  where $b' \neq b'' \in \arg\max_{b \in [K^{\mathsf{B}}]} U^{\mathsf{B}}(a', b)$, if such an action b'' exists, otherwise

  $$\mathsf{BRD}(t+1) = \mathsf{BRD}(t).$$

Best-response dynamics, continued

- If for some $\hat{t}$,

$$\mathrm{BRD}(\hat{t}) = \mathrm{BRD}(\hat{t}+1) = \mathrm{BRD}(\hat{t}+2) = (a^*, b^*),$$

  then $\mathrm{BRD}(t) = (a^*, b^*)$ for all $t \geq \hat{t}$ and (a^*, b^*) is a PNE of the game.
- The algorithm stops when it visits an action profile for the second time.
- If this profile is the same as the one visited at the previous time, then a PNE has been reached.

Traps

Definition

A *trap* is a finite set $\mathcal{T}$ of action profiles such that

(a) $|\mathcal{T}| \geq 2$,

(b) if $\mathsf{BRD}(t) \in \mathcal{T}$, then $\mathsf{BRD}(t+1) \in \mathcal{T}$;

(c) for every $(a, b) \in \mathcal{T}$, there exists t such that $\mathsf{BRD}(t + k|\mathcal{T}|) = (a, b)$, for every $k \in \mathbb{N}$.

Traps and pure Nash equilibria

- For every trap $\mathcal{T}$ we have $|\mathcal{T}| \geq 4$ and $|\mathcal{T}|$ even.
- For every game (U^A, U^B) and every initial profile (a, b), the BRD eventually visits a PNE or a trap in finite time, say τ.
- Even if the game admits PNE, there is no guarantee that a BRD reaches one of them; it could cycle over a trap.
- If the game is a potential game, then a BRD always reaches a PNE.

Interpolating i.i.d. and potential random games

- Each entry in $\boldsymbol{U}$ has a continuous distribution function F.
- Without loss of generality, take F uniform distribution on $[0, 1]$.
- Start with $(\boldsymbol{U}^{\mathsf{A}}, \boldsymbol{U}^{\mathsf{B}})$, where all the entries are i.i.d. with distribution F.
- Then, for each action profile (a, b), with probability p replace $U^{\mathsf{B}}(a, b)$ with $U^{\mathsf{A}}(a, b)$.
- For every pair (a, b),
  - with probability $1 - p$, the random payoffs $U^{\mathsf{A}}(a, b)$ and $U^{\mathsf{B}}(a, b)$ are independent,
  - with probability p, we have $U^{\mathsf{A}}(a, b) = U^{\mathsf{B}}(a, b)$.

Interpolation, continued

- The larger p, the closer the game is to a potential game.
- The smaller p, the closer the game is to a random game with i.i.d. payoffs.
- The game whose payoff bimatrix is obtained as above will be denoted by $\boldsymbol{U}(p)$.
- The analysis of this class of games is quite complicated for fixed $K^{\mathsf{A}}, K^{\mathsf{B}}$.
- We take an asymptotic approach, letting the number of actions grow.
- We consider a sequence $(\boldsymbol{U}_k)_{k \in \mathbb{N}}$ of payoff bimatrices, where the numbers of actions in game $\boldsymbol{U}_k$ are $K_k^{\mathsf{A}}, K_k^{\mathsf{B}}$, and these two integer sequences are increasing in k and diverge to ∞.
- We allow the number of actions of the two players to diverge at different speeds.

Number of pure Nash equilibria

Theorem

If W is the number of PNE in the game $\boldsymbol{U}(p)$, then

$$\mathrm{E}[W] = p\frac{K^{\mathsf{A}}K^{\mathsf{B}}}{K^{\mathsf{A}} + K^{\mathsf{B}} - 1} + (1 - p).$$

Asymptotics

Corollary

If $K_k^{\mathsf{B}} = \alpha_k K_k^{\mathsf{A}}$*, with* $\alpha_k \to \alpha$*, then*

$$\frac{W_k}{K_k^{\mathsf{A}}} \xrightarrow{\mathsf{P}} \frac{\alpha}{\alpha+1} p.$$

In particular, if $K_k^{\mathsf{A}} = K_k^{\mathsf{B}} = K_k$*, then*

$$\frac{W_k}{K_k} \xrightarrow{\mathsf{P}} \frac{p}{2}.$$

More generally

- $$h_k = \omega(g_k) \quad \text{if} \quad \lim_{k \to \infty} \frac{h_k}{g_k} = \infty,$$

Theorem

If $p_k = \omega(1/\sqrt{K_k})$, *then*

$$\frac{K_k^{\mathsf{A}} + K_k^{\mathsf{B}}}{p_k K_k^{\mathsf{A}} K_k^{\mathsf{B}}} W_k \xrightarrow{\mathsf{P}} 1.$$

BRD in potential games

- NE_k is the (random) set of PNE in the game $\boldsymbol{U}_k(p_k)$.

Theorem

If $p_k = 1$ and $\tau_k^{\mathsf{NE}} := \min\{t\colon\ \mathsf{BRD}_k(t) \in \mathsf{NE}_k\}$ is the first time the process $\mathsf{BRD}_k(t)$ visits a PNE, then

$$\mathsf{P}\big(\tau_k^{\mathsf{NE}} > t\big) = \prod_{j=0}^{t}(1 - q_{j,k}),$$

continued

- $$q_{t,k} := \mathrm{P}\big(\tau_k^{\mathrm{NE}} = t \mid \tau_k^{\mathrm{NE}} \geq t\big) \quad \text{for all } t \in \{0, \dots, 2K_k - 2\}.$$

- $$q_{0,k} = \frac{1}{K_k^{\mathrm{A}} + K_k^{\mathrm{B}} - 1},$$
  $$q_{1,k} = \frac{K_k^{\mathrm{A}} - 1}{K_k^{\mathrm{A}} + K_k^{\mathrm{B}} - 2},$$

  and, for $t \geq 2$,

  $$q_{t,k} = \frac{r_k(t-1)}{r_k(t)},$$

  where

  $$r_k(t) := \left\lceil \frac{t+1}{2} \right\rceil K_k^{\mathrm{A}} + \left\lfloor \frac{t+1}{2} \right\rfloor K_k^{\mathrm{B}} - \left\lfloor \frac{t+1}{2} \right\rfloor \left\lceil \frac{t+1}{2} \right\rceil, \quad \text{for } t \geq 1.$$

Mean and variance

Theorem
If $p_k = 1$ *and* $K_k^{\mathsf{A}} = K_k^{\mathsf{B}}$, *for every* $k \in \mathbb{N}$, *then*

$$\lim_{k\to\infty} \mathsf{E}\big[\tau_k^{\mathsf{NE}}\big] = \mathsf{e} - 1$$

$$\lim_{k\to\infty} \mathsf{Var}\big[\tau_k^{\mathsf{NE}}\big] \approx 0.767.$$

General case

- Amiet et al. (2021) showed that, when $p = 0$ and $K_k^{\mathsf{A}} = K_k^{\mathsf{B}}$, with high probability the BRD does not converge to the set NE_k.
- A tiny bit of correlation in the players payoffs, namely $p_k > 0$, is enough to dramatically change this phenomenon.

Theorem

Fix a sequence p_k. If

$$\lim_{k\to\infty} \frac{\log(p_k)}{\log(K_k^{\mathsf{A}} \wedge K_k^{\mathsf{B}})} = 0,$$

then

$$\lim_{k\to\infty} \mathsf{P}\big(\tau_k^{\mathsf{NE}} < \infty\big) = 1.$$

- If p_k is not too close to zero, an equilibrium will be found in time that is of order $p_k^{-\mathcal{O}(1)}$.

Fixed p

- If $p_k = p > 0$, for all $k \in \mathbb{N}$, then a sufficiently large but constant time is enough to find an equilibrium with high probability.

Corollary

Let $p_k = p \in (0,1)$*, for all* $k \in \mathbb{N}$*, and* $K_k^{\mathsf{A}} \wedge K_k^{\mathsf{B}} \to \infty$*. Then, for all* $\varepsilon > 0$*, there exists some* $T = T(\varepsilon)$ *such that*

$$\lim_{k\to\infty} \mathsf{P}\big(\tau_k^{\mathsf{NE}} < T\big) > 1 - \varepsilon.$$

Take-home

- We proposed a parametric class of probability measures on a space of finite normal-form games that interpolates between the extreme cases of i.i.d. payoffs and random potential games.
- We showed that the behavior of the i.i.d. case is different from all the other cases both in terms of number of pure Nash equilibria and of best-response dynamics.

Open problems

- Up to now the literature has analyzed the cases of many players and two actions or two players and many actions.
- The two cases required different tools.
- It would be nice to find the proper tools to analyze the general case of many players and many actions.
- The real challenge is to develop a general paradigm to choose a suitable probability measure on a space of games.

Thank you!

Relevant reference

- Redelmeier DA, Katz J, Kahneman D. Memories of colonoscopy: a randomized trial. *Pain*. 2003 Jul;104(1-2):187-94. doi: 10.1016/s0304-3959(03)00003-4. PMID: 12855328.