Hydrodynamic linear stability of the 2D bluff-body wake through modal analysis and initial-value problem formulation

Stefania Scarsoglio¹
Daniela Tordella² William O. Criminale³

¹Department of Hydraulics, Politecnico di Torino
²Department of Aeronautics and Space Engineering, Politecnico di Torino
³Department of Applied Mathematics, University of Washington

DICAT, Università di Genova
May 6th, 2010
Outline

1. Introduction
2. Physical Problem
Outline

1. Introduction
2. Physical Problem
3. Streamwise Entrainment Evolution
Outline

1. Introduction
2. Physical Problem
3. Streamwise Entrainment Evolution
4. Normal Mode Analysis
Outline

1. Introduction
2. Physical Problem
3. Streamwise Entrainment Evolution
4. Normal Mode Analysis
5. Transient and Long-Term Behavior of Small 3D Perturbations

Multiscale analysis for the stability of long 3D waves

Conclusions
Outline

1. Introduction
2. Physical Problem
3. Streamwise Entrainment Evolution
4. Normal Mode Analysis
5. Transient and Long-Term Behavior of Small 3D Perturbations
6. Multiscale analysis for the stability of long 3D waves
7. Conclusions
Outline

1. Introduction
2. Physical Problem
3. Streamwise Entrainment Evolution
4. Normal Mode Analysis
5. Transient and Long-Term Behavior of Small 3D Perturbations
6. Multiscale analysis for the stability of long 3D waves
7. Conclusions
Linear stability analysis of the 2D bluff-body wake

**Stability analysis**

- Hydrodynamics stability is important in different fields (aerodynamics, oceanography, environmental sciences, etc);
Linear stability analysis of the 2D bluff-body wake

- **Stability analysis**
  - Hydrodynamics stability is important in different fields (aerodynamics, oceanography, environmental sciences, etc);
  - To understand the reasons for the breakdown of laminar flow;
Linear stability analysis of the 2D bluff-body wake

- **Stability analysis**
  - Hydrodynamics stability is important in different fields (aerodynamics, oceanography, environmental sciences, etc);
  - To understand the reasons for the breakdown of laminar flow;
  - To predict the transition to turbulence.
Linear stability analysis of the 2D bluff-body wake

- **Stability analysis**
  - Hydrodynamics stability is important in different fields (aerodynamics, oceanography, environmental sciences, etc);
  - To understand the reasons for the breakdown of laminar flow;
  - To predict the transition to turbulence.

- **Modal theory**
  - Flow asymptotically stable or unstable;
Linear stability analysis of the 2D bluff-body wake

- **Stability analysis**
  - Hydrodynamics stability is important in different fields (aerodynamics, oceanography, environmental sciences, etc);
  - To understand the reasons for the breakdown of laminar flow;
  - To predict the transition to turbulence.

- **Modal theory**
  - Flow asymptotically stable or unstable;
  - Discrete spectrum (not complete for unbounded flows);
Linear stability analysis of the 2D bluff-body wake

- **Stability analysis**
  - Hydrodynamics stability is important in different fields (aerodynamics, oceanography, environmental sciences, etc);
  - To understand the reasons for the breakdown of laminar flow;
  - To predict the transition to turbulence.

- **Modal theory**
  - Flow asymptotically stable or unstable;
  - Discrete spectrum (not complete for unbounded flows);
  - Convective and absolute instability.
Introduction
Physical Problem
Streamwise Entrainment Evolution
Normal Mode Analysis
Transient and Long-Term Behavior of Small 3D Perturbations
Multiscale analysis for the stability of long 3D waves
Conclusions

Linear stability analysis of the 2D bluff-body wake

- **Stability analysis**
  - Hydrodynamics stability is important in different fields (aerodynamics, oceanography, environmental sciences, etc);
  - To understand the reasons for the breakdown of laminar flow;
  - To predict the transition to turbulence.

- **Modal theory**
  - Flow asymptotically stable or unstable;
  - Discrete spectrum (not complete for unbounded flows);
  - Convective and absolute instability.

- **Initial-value problem**
  - Temporal evolution of arbitrary disturbances;
Linear stability analysis of the 2D bluff-body wake

- **Stability analysis**
  - Hydrodynamics stability is important in different fields (aerodynamics, oceanography, environmental sciences, etc);
  - To understand the reasons for the breakdown of laminar flow;
  - To predict the transition to turbulence.

- **Modal theory**
  - Flow asymptotically stable or unstable;
  - Discrete spectrum (not complete for unbounded flows);
  - Convective and absolute instability.

- **Initial-value problem**
  - Temporal evolution of arbitrary disturbances;
  - Importance of the transient growth (e.g. *by-pass transition*);
Linear stability analysis of the 2D bluff-body wake

- **Stability analysis**
  - Hydrodynamics stability is important in different fields (aerodynamics, oceanography, environmental sciences, etc);
  - To understand the reasons for the breakdown of laminar flow;
  - To predict the transition to turbulence.

- **Modal theory**
  - Flow asymptotically stable or unstable;
  - Discrete spectrum (not complete for unbounded flows);
  - Convective and absolute instability.

- **Initial-value problem**
  - Temporal evolution of arbitrary disturbances;
  - Importance of the transient growth (e.g. by-pass transition);
  - Aim to understand the cause of any possible instability in terms of the underlying physics.
The two-dimensional bluff-body wake

Flow behind a circular cylinder:
The two-dimensional bluff-body wake

- Flow behind a circular cylinder:
  ⇒ Steady, incompressible and viscous;
The two-dimensional bluff-body wake

- Flow behind a circular cylinder:  
  ⇒ Steady, incompressible and viscous;

- Approximation of 2D asymptotic Navier-Stokes expansions (Belen & Tordella, *Phys. Fluids*, 2003), $20 \leq Re \leq 100$. 

S. Scarsoglio

DICAT, Università di Genova
The two-dimensional bluff-body wake

- Flow behind a circular cylinder:
  ⇒ **Steady, incompressible and viscous**;

- Approximation of 2D asymptotic Navier-Stokes expansions (Belen & Tordella, *Phys. Fluids*, 2003), $20 \leq Re \leq 100$. 

![Graph showing flow profiles](image)
The two-dimensional bluff-body wake
Velocity Flow Rate Defect and Entrainment

- Defect of the volumetric flow rate $D$:

$$D(x) = \int_{-\infty}^{+\infty} (1 - U(x, y)) dy$$

The first $R_{cr}$ as a possible measure of the entrainment length
Velocity Flow Rate Defect and Entrainment

• **Defect of the volumetric flow rate** $D$:

$$D(x) = \int_{-\infty}^{+\infty} (1 - U(x, y)) dy$$

• **Entrainment** $E$ takes into account the variation of the defect of the volumetric flow rate in the streamwise direction:

$$E(x) = \left| \frac{dD(x)}{dx} \right|$$

The first $R_{cr}$ as a possible measure of the entrainment length

### Results

(a) Paranthoen et al. (1999) $Re=53.3$ (unsteady flow)

Takami & Keller (1969) $Re=40$

Kovasznay (1948) $Re=56$ (unsteady flow)

(b) Paranthoen et al. (1999) $Re=53.3$ (unsteady flow)

Takami & Keller (1969) $Re=40$

Kovasznay (1948) $Re=56$ (unsteady flow)

(c) $x=8$

$x=10$

$x=20$

$x=40$

$x=80$

(d) $x=8$

$x=10$

$x=20$

$x=40$

$x=80$

S. Scarsoglio

DICAT, Università di Genova
Normal Mode Theory

- The linearized perturbative equation in terms of stream function $\psi(x, y, t)$ is

$$\partial_t \nabla^2 \psi + (\partial_x \nabla^2 \psi) \psi_y + \psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \psi) \psi_x - \psi_x \partial_y \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi$$
Normal Mode Theory

The linearized perturbative equation in terms of stream function $\psi(x, y, t)$ is

$$\partial_t \nabla^2 \psi + (\partial_x \nabla^2 \psi) \psi_y + \psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \psi) \psi_x - \psi_x \partial_y \nabla^2 \psi = \frac{1}{\text{Re}} \nabla^4 \psi$$

- Normal mode hypothesis $\Rightarrow \psi(x, y, t) = \varphi(x, y, t) e^{i(h_0 x - \sigma_0 t)}$
Normal Mode Theory

- The linearized perturbative equation in terms of stream function \( \psi(x, y, t) \) is

\[
\partial_t \nabla^2 \psi + (\partial_x \nabla^2 \psi) \psi_y + \psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \psi) \psi_x - \psi_x \partial_y \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi
\]

- **Normal mode hypothesis** \( \Rightarrow \psi(x, y, t) = \varphi(x, y, t) e^{i(h_0 x - \sigma_0 t)} \)
  - \( \varphi(x, y, t) \) complex eigenfunction;
Normal Mode Theory

- The linearized perturbative equation in terms of stream function $\psi(x, y, t)$ is

$$\partial_t \nabla^2 \psi + (\partial_x \nabla^2 \psi) \psi_y + \psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \psi) \psi_x - \psi_x \partial_y \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi$$

- **Normal mode hypothesis** $\Rightarrow$ $\psi(x, y, t) = \varphi(x, y, t) e^{i(h_0 x - \sigma_0 t)}$
  - $\varphi(x, y, t)$ complex eigenfunction;
  - $h_0 = k_0 + i s_0$ complex wavenumber ($k_0$ wavenumber, $s_0$ spatial growth rate);
Normal Mode Theory

- The linearized perturbative equation in terms of stream function $\psi(x, y, t)$ is

$$\partial_t \nabla^2 \psi + (\partial_x \nabla^2 \psi) \psi_y + \psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \psi) \psi_x - \psi_x \partial_y \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi$$

- **Normal mode hypothesis** $\Rightarrow \psi(x, y, t) = \varphi(x, y, t) e^{i(h_0 x - \sigma_0 t)}$
  - $\varphi(x, y, t)$ complex eigenfunction;
  - $h_0 = k_0 + i s_0$ complex wavenumber ($k_0$ wavenumber, $s_0$ spatial growth rate);
  - $\sigma_0 = \omega_0 + i r_0$ complex frequency ($\omega_0$ frequency, $r_0$ temporal growth rate);
Normal Mode Theory

- The linearized perturbative equation in terms of stream function $\psi(x, y, t)$ is

$$\partial_t \nabla^2 \psi + (\partial_x \nabla^2 \psi) \psi_y + \psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \psi) \psi_x - \psi_x \partial_y \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi$$

- **Normal mode hypothesis** $\Rightarrow \psi(x, y, t) = \varphi(x, y, t) e^{i(h_0 x - \sigma_0 t)}$
  - $\varphi(x, y, t)$ complex eigenfunction;
  - $h_0 = k_0 + is_0$ complex wavenumber ($k_0$ wavenumber, $s_0$ spatial growth rate);
  - $\sigma_0 = \omega_0 + ir_0$ complex frequency ($\omega_0$ frequency, $r_0$ temporal growth rate);

- **Convective instability**: $r_0 < 0$ for all modes, $s_0 < 0$ for at least one mode.
Normal Mode Theory

- The linearized perturbative equation in terms of stream function $\psi(x, y, t)$ is

$$\partial_t \nabla^2 \psi + (\partial_x \nabla^2 \psi) \psi_y + \psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \psi) \psi_x - \psi_x \partial_y \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi$$

- **Normal mode hypothesis** $\Rightarrow \psi(x, y, t) = \varphi(x, y, t) e^{i(h_0 x - \sigma_0 t)}$
  - $\varphi(x, y, t)$ complex eigenfunction;
  - $h_0 = k_0 + is_0$ complex wavenumber ($k_0$ wavenumber, $s_0$ spatial growth rate);
  - $\sigma_0 = \omega_0 + ir_0$ complex frequency ($\omega_0$ frequency, $r_0$ temporal growth rate);

- **Convective instability**: $r_0 < 0$ for all modes, $s_0 < 0$ for at least one mode.

- **Absolute instability**: $r_0 > 0$, $\partial \sigma_0 / \partial h_0 = 0$ for at least one mode.
Stability analysis through multiscale approach

- Slow variables: \( x_1 = \epsilon x, \ t_1 = \epsilon t, \ \epsilon = 1/Re. \)
Stability analysis through multiscale approach

- **Slow variables**: \( x_1 = \epsilon x, \ t_1 = \epsilon t, \ \epsilon = 1/Re. \)
- **Hypothesis**: \( \psi(x, y, t) \) and \( \Psi(x, y, t) \) are expansions in terms of \( \epsilon \): (ODE dependent on \( \varphi_0 \)) + \( \epsilon \) (ODE dependent on \( \varphi_0, \varphi_1 \)) + \( O(\epsilon^2) \)
Stability analysis through multiscale approach

- Slow variables: \( x_1 = \epsilon x, t_1 = \epsilon t, \epsilon = 1/Re \).
- **Hypothesis:** \( \psi(x, y, t) \) and \( \Psi(x, y, t) \) are expansions in terms of \( \epsilon \):
  (ODE dependent on \( \varphi_0 \)) + \( \epsilon \) (ODE dependent on \( \varphi_0, \varphi_1 \)) + \( O(\epsilon^2) \)
- **Order zero:** homogeneous Orr-Sommerfeld equation
  \[
  \mathcal{A} \varphi_0 = \sigma_0 \mathcal{B} \varphi_0 \quad \mathcal{A} = (\partial_y^2 - h_0^2)^2 - ih_0 Re[u_0(\partial_y^2 - h_0^2) - \partial_y^2 u_0] \\
  \varphi_0 \rightarrow 0, |y| \rightarrow \infty \quad \mathcal{B} = -iRe(\partial_y^2 - h_0^2) \\
  \partial_y \varphi_0 \rightarrow 0, |y| \rightarrow \infty
  \]
  \( \Rightarrow \) eigenfunctions \( \varphi_0 \) and a discrete set of eigenvalues \( \sigma_{0n} \).
Stability analysis through multiscale approach

- **Slow variables:** $x_1 = \epsilon x$, $t_1 = \epsilon t$, $\epsilon = 1/Re$.
- **Hypothesis:** $\psi(x, y, t)$ and $\Psi(x, y, t)$ are expansions in terms of $\epsilon$: 
  $$(\text{ODE dependent on } \varphi_0) + \epsilon (\text{ODE dependent on } \varphi_0, \varphi_1) + O(\epsilon^2)$$
- **Order zero:** homogeneous Orr-Sommerfeld equation
  $$A \varphi_0 = \sigma_0 B \varphi_0$$
  $$\varphi_0 \to 0, |y| \to \infty$$
  $$\partial_y \varphi_0 \to 0, |y| \to \infty$$
  $$B = -iRe(\partial_y^2 - h_0^2)$$
  $$\Rightarrow$$ eigenfunctions $\varphi_0$ and a discrete set of eigenvalues $\sigma_{0n}$.
- **First order:** Non homogeneous Orr-Sommerfeld equation
  $$A \varphi_1 = \sigma_0 B \varphi_1 + M \varphi_0$$
  $$\varphi_1 \to 0, |y| \to \infty$$
  $$\partial_y \varphi_1 \to 0, |y| \to \infty$$
  $$M = \left[ Re(2h_0\sigma_0 - 3h_0^2u_0 - \partial_y^2u_0) + 4ih_0^3 \right] \partial_{x_1}$$
  $$+ (Reu_0 - 4ih_0)\partial_{x_1, yy} + Rev_1(\partial_y^3 - h_0^2\partial_y) + Re\partial_y^2v_1\partial_y$$
  $$+ ih_0 Re \left[ u_1(\partial_y^2 - h_0^2) - \partial_y^2u_1 \right] + Re(\partial_y^2 - h_0^2)\partial_{t_1}$$
Perturbative hypothesis: saddle point sequence

- For fixed values of $x$ and $Re$, the saddle points $(h_{0s}, \sigma_{0s})$ of the dispersion relation $\sigma_0 = \sigma_0(h_0, x, Re)$ satisfy $\partial \sigma_0 / \partial h_0 = 0$;
Perturbative hypothesis: saddle point sequence

For fixed values of $x$ and $Re$, the saddle points $(h_{0s}, \sigma_{0s})$ of the dispersion relation $\sigma_0 = \sigma_0(h_0, x, Re)$ satisfy $\partial \sigma_0 / \partial h_0 = 0$;

$Re = 35, x = 4$. Level curves, $\omega_0 =$const (thin curves), $r_0 =$const (thick curves).
Perturbative hypothesis: saddle point sequence

\[ \text{Re} = 35, \ x = 4. \ \omega_0(k_0, s_0), \ r_0(k_0, s_0). \]
Instability Characteristics

(a) Streamwise wave number evolution for different Reynolds numbers:
- $k_0$ and $k$ for $Re=35$, $Re=50$, $Re=100$

(b) Growth rate $s$ as a function of streamwise location $x$:
- $s_0$ and $s$ for $Re=35$, $Re=50$, $Re=100$

(c) Angular frequency $\omega$:
- $\omega_0$ and $\omega$ for $Re=35$, $Re=50$, $Re=100$

(d) Swirl number $r$:
- $r_0$ and $r$ for $Re=35$, $Re=50$, $Re=100$

S. Scarsoglio
DICAT, Università di Genova
Global Pulsation


Formulation

- Linear three-dimensional perturbative equations in terms of velocity and vorticity (Criminale & Drazin, *Stud. Applied Math.*, 1990);
Formulation

- Linear three-dimensional perturbative equations in terms of velocity and vorticity (Criminale & Drazin, *Stud. Applied Math.*, 1990);
- Base flow parametric in $x$ and $Re \Rightarrow U(y; x_0, Re)$;
Formulation

- Linear three-dimensional perturbative equations in terms of velocity and vorticity (Criminale & Drazin, *Stud. Applied Math.*, 1990);
- Base flow parametric in $x$ and $Re \Rightarrow U(y; x_0, Re)$;
- Laplace-Fourier transform in $x$ and $z$ directions, $\alpha$ complex, $\gamma$ real;
Formulation

- Linear three-dimensional perturbative equations in terms of velocity and vorticity (Criminale & Drazin, *Stud. Applied Math.*, 1990);
- Base flow parametric in $x$ and $Re \Rightarrow U(y; x_0, Re)$;
- Laplace-Fourier transform in $x$ and $z$ directions, $\alpha$ complex, $\gamma$ real;

\[ \gamma = \text{transversal wavenumber} \]
\[ \alpha_r = \text{longitudinal wavenumber} \]
\[ \phi = \text{angle of obliquity} \]
\[ k = \text{polar wavenumber} \]
\[ \alpha_i = \text{spatial damping rate} \]
Perturbative equations

- Perturbative linearized system:

\[ \frac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{v} = \hat{\Gamma} \]

\[ \frac{\partial \hat{\Gamma}}{\partial t} = (i\alpha_r - \alpha_i)(\frac{d^2 U}{dy^2} \hat{v} - U\hat{\Gamma}) + \frac{1}{Re} \left[ \frac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\Gamma} \right] \]

\[ \frac{\partial \hat{\omega}_y}{\partial t} = -(i\alpha_r - \alpha_i)U\hat{\omega}_y - i\gamma \frac{dU}{dy} \hat{v} + \frac{1}{Re} \left[ \frac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\omega}_y \right] \]
Perturbative equations

- Perturbative linearized system:

\[
\frac{\partial^2 \hat{v}}{\partial y^2} - \left( k^2 - \alpha_i^2 + 2i \alpha_r \alpha_i \right) \hat{v} = \hat{\Gamma}
\]

\[
\frac{\partial \hat{\Gamma}}{\partial t} = (i \alpha_r - \alpha_i) \left( \frac{d^2 U}{dy^2} \hat{v} - U \hat{\Gamma} \right) + \frac{1}{Re} \left[ \frac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i \alpha_r \alpha_i) \hat{\Gamma} \right]
\]

\[
\frac{\partial \hat{\omega}_y}{\partial t} = -(i \alpha_r - \alpha_i) U \hat{\omega}_y - i \gamma \frac{dU}{dy} \hat{v} + \frac{1}{Re} \left[ \frac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i \alpha_r \alpha_i) \hat{\omega}_y \right]
\]

The transversal velocity and vorticity components are \( \hat{v} \) and \( \hat{\omega}_y \) respectively, \( \hat{\Gamma} \) is defined as \( \hat{\Gamma} = \partial_x \hat{\omega}_z - \partial_z \hat{\omega}_x \).
Perturbative equations

- Perturbative linearized system:

\[
\begin{align*}
\frac{\partial^2 \hat{v}}{\partial y^2} & = (k^2 - \alpha_i^2 + 2i\alpha r \alpha_i)\hat{v} = \hat{\Gamma} \\
\frac{\partial \hat{\Gamma}}{\partial t} & = (i\alpha_r - \alpha_i)(\frac{d^2 U}{dy^2} \hat{v} - U\hat{\Gamma}) + \frac{1}{Re} \left[ \frac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha r \alpha_i)\hat{\Gamma} \right] \\
\frac{\partial \hat{\omega}_y}{\partial t} & = -(i\alpha_r - \alpha_i)U\hat{\omega}_y - i\gamma \frac{dU}{dy} \hat{v} + \frac{1}{Re} \left[ \frac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha r \alpha_i)\hat{\omega}_y \right]
\end{align*}
\]

The transversal velocity and vorticity components are \( \hat{v} \) and \( \hat{\omega}_y \) respectively, \( \hat{\Gamma} \) is defined as \( \hat{\Gamma} = \partial_x \hat{\omega}_z - \partial_z \hat{\omega}_x \).

- Initial conditions:
  - \( \hat{\omega}_y(0, y) = 0; \)
Perturbative equations

- **Perturbative linearized system:**

\[
\begin{align*}
\frac{\partial^2 \hat{v}}{\partial y^2} &\quad - (k^2 - \alpha_i^2 + 2i \alpha_r \alpha_i) \hat{v} = \hat{\Gamma} \\
\frac{\partial \hat{\Gamma}}{\partial t} &\quad = (i \alpha_r - \alpha_i) \left( \frac{d^2 U}{dy^2} \hat{v} - U \hat{\Gamma} \right) + \frac{1}{\text{Re}} \left[ \frac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i \alpha_r \alpha_i) \hat{\Gamma} \right] \\
\frac{\partial \hat{\omega}_y}{\partial t} &\quad = -(i \alpha_r - \alpha_i) U \hat{\omega}_y - i \gamma \frac{dU}{dy} \hat{v} + \frac{1}{\text{Re}} \left[ \frac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i \alpha_r \alpha_i) \hat{\omega}_y \right]
\end{align*}
\]

The transversal velocity and vorticity components are \( \hat{v} \) and \( \hat{\omega}_y \) respectively, \( \hat{\Gamma} \) is defined as \( \hat{\Gamma} = \partial_x \hat{\omega}_z - \partial_z \hat{\omega}_x \).

- **Initial conditions:**
  - \( \hat{\omega}_y(0, y) = 0; \)
  - \( \hat{\Gamma}(0, y) = e^{-y^2} \sin(y) \) or \( \hat{\Gamma}(0, y) = e^{-y^2} \cos(y); \)
Perturbative equations

- **Perturbative linearized system:**

\[
\begin{align*}
\frac{\partial^2 \hat{v}}{\partial y^2} &- (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{v} = \hat{\Gamma} \\
\frac{\partial \hat{\Gamma}}{\partial t} &= (i\alpha_r - \alpha_i)(\frac{d^2 U}{dy^2} \hat{v} - U\hat{\Gamma}) + \frac{1}{Re} [\frac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\Gamma}] \\
\frac{\partial \hat{\omega}_y}{\partial t} &= -(i\alpha_r - \alpha_i)U\hat{\omega}_y - i\gamma \frac{dU}{dy} \hat{v} + \frac{1}{Re} [\frac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\omega}_y]
\end{align*}
\]

The transversal velocity and vorticity components are \( \hat{v} \) and \( \hat{\omega}_y \) respectively, \( \hat{\Gamma} \) is defined as \( \hat{\Gamma} = \partial_x \hat{\omega}_z - \partial_z \hat{\omega}_x \).

- **Initial conditions:**
  - \( \hat{\omega}_y(0, y) = 0; \)
  - \( \hat{\Gamma}(0, y) = e^{-y^2}\sin(y) \) or \( \hat{\Gamma}(0, y) = e^{-y^2}\cos(y); \)

- **Boundary conditions:** \( (\hat{u}, \hat{v}, \hat{w}) \to 0 \) as \( y \to \infty \).
Measure of the Growth

- Kinetic energy density $e$:

$$
e(t; \alpha, \gamma) = \frac{1}{2} \frac{1}{2y_d} \int_{-y_d}^{+y_d} (|\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2) dy
= \frac{1}{2} \frac{1}{2y_d} \frac{1}{|\alpha^2 + \gamma^2|} \int_{-y_d}^{+y_d} \left( \frac{\partial \hat{v}}{\partial y} \right)^2 + |\hat{\alpha}^2 + \gamma^2| |\hat{v}|^2 + |\hat{\omega}_y|^2) dy
$$

S. Scarsoglio
DICAT, Università di Genova
Measure of the Growth

- **Kinetic energy density** $e$:

$$e(t; \alpha, \gamma) = \frac{1}{2} \frac{1}{2y_d} \int_{-y_d}^{+y_d} (|\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2) dy$$

$$= \frac{1}{2} \frac{1}{2y_d} \frac{1}{|\alpha^2 + \gamma^2|} \int_{-y_d}^{+y_d} (|\frac{\partial \hat{v}}{\partial y}|^2 + |\alpha^2 + \gamma^2| |\hat{v}|^2 + |\hat{\omega}_y|^2) dy$$

- **Amplification factor** $G$:

$$G(t; \alpha, \gamma) = \frac{e(t; \alpha, \gamma)}{e(t = 0; \alpha, \gamma)}$$
Measure of the Growth

- Temporal growth rate $r$ (Lasseigne et al., *J. Fluid Mech.*, 1999):

$$r(t; \alpha, \gamma) = \frac{\log|e(t; \alpha, \gamma)|}{2t}, \quad t > 0$$
Measure of the Growth

- **Temporal growth rate** $r$ (Lasseigne et al., *J. Fluid Mech.*, 1999):
  \[
  r(t; \alpha, \gamma) = \frac{\log|e(t; \alpha, \gamma)|}{2t}, \quad t > 0
  \]

- **Angular frequency (pulsation)** $\omega$ (Whitham, 1974):
  \[
  \omega(t; \alpha, \gamma) = \frac{d\varphi(t)}{dt}, \quad \varphi \text{ time phase}
  \]
Effect of $\alpha_i$ and $k$

Effect of the symmetry of the perturbation

(a) \( x_0 = 10 \) (intermediate) \( \tau_{\text{inter}} = 35 \)
\( x_0 = 50 \) (far) \( \tau_{\text{far}} = 100 \)

\( \alpha_i = 0.02 \)
\( \phi = \pi/4 \)

(b) \( x_0 = 10 \) (intermediate)
\( \alpha_i = 0.02 \)
\( \phi = \pi/4 \)

\( \tau_{\text{far}} = 100 \)
\( \tau_{\text{inter}} = 35 \)

\( \Re = 100 \)
\( k = 0.6 \)
Effect of the symmetry of the perturbation

(a) $G(t)$ for $x_0=10$ (intermediate) and $x_0=50$ (far)
- Asymmetric input
  - $	au_{\text{inter}} = 35$
  - $	au_{\text{far}} = 100$

(b) $G(t)$ for $x_0=10$ (intermediate) and $x_0=50$ (far)
- Symmetric input
  - $	au_{\text{inter}} = 35$
  - $	au_{\text{far}} = 100$

(c) $\omega(t)$
- Asymmetric input
- Symmetric input

Parameters:
- $Re=100$
- $k=0.6$
- $\alpha_i=0.02$
- $\phi=\pi/4$

S. Scarsoglio
DICAT, Università di Genova
Effect of the symmetry of the perturbation

3D Visualization

(a) $x_0=10$ (intermediate) $x_0=50$ (far)

$\tau_{\text{far}} = 100$

$\tau_{\text{inter}} = 35$

asymmetric input

$G$

$10^0$

$10^1$

$10^2$

$10^3$

$10^4$

$t$

$0$

$100$

$200$

$300$

$400$

$500$

Re=100

$k=0.6$

$\alpha_i=0.02$

$\phi=\pi/4$

(b) $x_0=10$ (intermediate) $x_0=50$ (far)

$G$

$10^{-5}$

$10^{-4}$

$10^{-3}$

$10^{-2}$

$10^{-1}$

$10^0$

$10^1$

$10^2$

$t$

$0$

$50$

$100$

$150$

$200$

$250$

symmetric input

Re=100

$k=0.6$

$\phi=\pi/4$

$\alpha_i=0.02$

(c)

$\omega$

$0.5$

$1$

$1.5$

$0$

$0.5$

$1$

$1.5$

$2$

$3$

$4$

$5$

$6$

$7$

$8$

$9$

$10$

$t$

$0$

$100$

$200$

$300$

$400$

$500$

Re=100

$k=0.6$

$\alpha_i=0.02$

$\phi=\pi/4$

$\omega_{\text{sym}}$

$\omega_{\text{asym}}$
Effect of $\phi$

- $\phi = 0$
- $\phi = \pi/2$

- $\tau_{\text{inter}} = 2.5$
- $x_0 = 10$
- $\text{Re} = 100$
- $\alpha_i = 0.02$
- $k = 0.7$

- Asymmetric input

Initial-Value Problem
Exploratory Analysis of the Transient Dynamics
Asymptotic State

S. Scarsoglio
DICAT, Università di Genova
Comparison with modal analysis and laboratory data
Angular frequency and temporal growth rate, $\alpha_i = 0.05$, $\phi = 0$, $x_0 = 10$.

Full linear problem

- Linearized 3D equations and Laplace-Fourier transform \((x, z)\);
Full linear problem

- Linearized 3D equations and Laplace-Fourier transform \((x, z)\);
- Base flow parametric in \(x\) and \(Re \Rightarrow (U(y; x_0, Re), V(y; x_0, Re))\);
Full linear problem

- Linearized 3D equations and Laplace-Fourier transform \((x, z)\);
- Base flow parametric in \(x\) and \(Re \Rightarrow (U(y; x_0, Re), V(y; x_0, Re))\);

\[
\begin{align*}
\frac{\partial^2 \hat{v}}{\partial y^2} & - (k^2 - \alpha_i^2 + 2ik\cos(\phi)\alpha_i)\hat{v} = \hat{\Gamma} \\
\frac{\partial \hat{\Gamma}}{\partial t} & = G\hat{\Gamma} + H\hat{v} + K\hat{\omega}_y \\
\frac{\partial \hat{\omega}_y}{\partial t} & = L\hat{\omega}_y + M\hat{v}
\end{align*}
\]
Full linear problem

- Linearized 3D equations and Laplace-Fourier transform \((x, z)\);
- Base flow parametric in \(x\) and \(Re \Rightarrow (U(y; x_0, Re), V(y; x_0, Re))\);

\[
\frac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2ik\cos(\phi)\alpha_i)\hat{v} = \hat{\Gamma}
\]

\[
\frac{\partial \hat{r}}{\partial t} = G\hat{r} + H\hat{v} + K\hat{\omega}_y
\]

\[
\frac{\partial \hat{\omega}_y}{\partial t} = L\hat{\omega}_y + M\hat{v}
\]

- \(G = G(y; x_0, k, \phi, \alpha_i, Re)\), and similarly \(H, K, L\) and \(M\), are ordinary differential operators.
Multiple scales hypothesis

- Regular perturbation scheme, $k \ll 1$:

\[
\begin{align*}
\hat{v} & = \hat{v}_0 + k\hat{v}_1 + k^2\hat{v}_2 + \cdots, \\
\hat{\Gamma} & = \hat{\Gamma}_0 + k\hat{\Gamma}_1 + k^2\hat{\Gamma}_2 + \cdots, \\
\hat{\omega}_y & = \hat{\omega}_{y0} + k\hat{\omega}_{y1} + k^2\hat{\omega}_{y2} + \cdots.
\end{align*}
\]
Multiple scales hypothesis

- Regular perturbation scheme, $k \ll 1$:
  
  \[
  \hat{\nu} = \hat{\nu}_0 + k\hat{\nu}_1 + k^2\hat{\nu}_2 + \cdots , \\
  \hat{\Gamma} = \hat{\Gamma}_0 + k\hat{\Gamma}_1 + k^2\hat{\Gamma}_2 + \cdots , \\
  \hat{\omega}_y = \hat{\omega}_{y0} + k\hat{\omega}_{y1} + k^2\hat{\omega}_{y2} + \cdots .
  \]

- Temporal scales: $t$, $\tau = kt$, $T = k^2t$;
Multiple scales hypothesis

- Regular perturbation scheme, $k \ll 1$:
  \[
  \hat{v} = \hat{v}_0 + k\hat{v}_1 + k^2\hat{v}_2 + \cdots, \\
  \hat{\Gamma} = \hat{\Gamma}_0 + k\hat{\Gamma}_1 + k^2\hat{\Gamma}_2 + \cdots, \\
  \hat{\omega}_y = \hat{\omega}_y0 + k\hat{\omega}_y1 + k^2\hat{\omega}_y2 + \cdots.
  \]

- Temporal scales: $t$, $\tau = kt$, $T = k^2t$;
- Spatial scales: $y$, $Y = ky$. 

S. Scarsoglio
DICAT, Università di Genova
Multiple scales equations up to $O(k)$

- **Order $O(1)$**

\[
\frac{\partial^2 \hat{v}_0}{\partial y^2} + \alpha_i^2 \hat{v}_0 = \hat{\Gamma}_0
\]

\[
\frac{\partial \hat{\Gamma}_0}{\partial t} - G_0 \hat{\Gamma}_0 - H_0 \hat{v}_0 = 0
\]

\[
\frac{\partial \hat{\omega}_y}{\partial t} - L_0 \hat{\omega}_y = 0
\]
Multiple scales equations up to $O(k)$

**Order $O(1)$**

\[
\frac{\partial^2 \hat{v}_0}{\partial y^2} + \alpha_i^2 \hat{v}_0 = \hat{\Gamma}_0 \\
\frac{\partial \hat{\Gamma}_0}{\partial t} - G_0 \hat{\Gamma}_0 - H_0 \hat{v}_0 = 0 \\
\frac{\partial \hat{\omega}_y}{\partial t} - L_0 \hat{\omega}_y = 0
\]

where $G_0 = G_0(y; x_0, \phi, \alpha_i, Re)$ and similarly for $H_0$ and $L_0$. 
Multiple scales equations up to $O(k)$

- **Order $O(k)$**

\[
\begin{align*}
\frac{\partial^2 \hat{v}_1}{\partial y^2} + \alpha_i^2 \hat{v}_1 &= -2 \frac{\partial^2 \hat{v}_0}{\partial y \partial Y} + 2 \text{icosec}(\phi) \alpha_i \hat{v}_0 + \hat{\Gamma}_1 \\
\frac{\partial \hat{\Gamma}_1}{\partial t} - G_0 \hat{\Gamma}_1 - H_0 \hat{v}_1 &= -\frac{\partial \hat{\Gamma}_0}{\partial \tau} + G_1 \hat{\Gamma}_0 + H_1 \hat{v}_0 + K_1 \hat{\omega}_y \\
\frac{\partial \hat{\omega}_y}{\partial t} - L_0 \hat{\omega}_y &= -\frac{\partial \hat{\omega}_y}{\partial \tau} + L_1 \hat{\omega}_y + M_1 \hat{v}_0
\end{align*}
\]
Multiple scales equations up to $O(k)$

- **Order $O(k)$**

\[
\frac{\partial^2 \hat{v}_1}{\partial y^2} + \alpha_i^2 \hat{v}_1 = -2 \frac{\partial^2 \hat{v}_0}{\partial y \partial Y} + 2i \cos(\phi) \alpha_i \hat{v}_0 + \hat{\Gamma}_1
\]

\[
\frac{\partial \hat{\Gamma}_1}{\partial t} - G_0 \hat{\Gamma}_1 - H_0 \hat{v}_1 = - \frac{\partial \hat{\Gamma}_0}{\partial \tau} + G_1 \hat{\Gamma}_0 + H_1 \hat{v}_0 + K_1 \hat{\omega}_y_0
\]

\[
\frac{\partial \hat{\omega}_y_1}{\partial t} - L_0 \hat{\omega}_y_1 = - \frac{\partial \hat{\omega}_y_0}{\partial \tau} + L_1 \hat{\omega}_y_0 + M_1 \hat{v}_0
\]

where $G_1 = G_1(y, Y; x_0, \phi, \alpha_i, Re)$ and similarly for $H_1, K_1, L_1$ and $M_1$. 
Effect of $\alpha_i$ and $k$

Effect of the symmetry of the perturbation

\[ G \] vs. \[ t \]

\[ x_0 = 10 \quad \alpha_i = 0.08 \]

\[ \text{Re} = 100 \quad k = 0.02 \quad \phi = \pi/2 \]

Comparison between multiscale and full problem results

S. Scarsoglio
DICAT, Università di Genova
Asymptotic state

- Temporal asymptotic values of the angular frequency $\omega$ and the temporal growth rate $r$.

![Graph showing temporal asymptotic values of $\omega$ and $r$.]
Conclusions

- **Modal analysis**
  - Synthetic perturbation hypothesis (saddle point sequence);
Conclusions

- **Modal analysis**
  - Synthetic perturbation hypothesis (saddle point sequence);
  - Absolute instability pockets in the intermediate wake ($Re = 50, 100$);
Conclusions

- **Modal analysis**
  - Synthetic perturbation hypothesis (saddle point sequence);
  - Absolute instability pockets in the intermediate wake ($Re = 50, 100$);
  - Frequency in good agreement with numerical and experimental data;
Conclusions

- **Modal analysis**
  - Synthetic perturbation hypothesis (saddle point sequence);
  - Absolute instability pockets in the intermediate wake \((Re = 50, 100)\);
  - Frequency in good agreement with numerical and experimental data;
  - *No information on the early time history of the perturbation*;
Conclusions

- **Modal analysis**
  - Synthetic perturbation hypothesis (saddle point sequence);
  - Absolute instability pockets in the intermediate wake ($Re = 50, 100$);
  - Frequency in good agreement with numerical and experimental data;
  - *No information on the early time history of the perturbation*;

- **Initial-value problem**
  - How the transient is affected by the perturbation: different growths of energy, length of the transient, variety of temporal scales;
Conclusions

- **Modal analysis**
  - Synthetic perturbation hypothesis (saddle point sequence);
  - Absolute instability pockets in the intermediate wake \((Re = 50, 100)\);
  - Frequency in good agreement with numerical and experimental data;
  - *No information on the early time history of the perturbation*;

- **Initial-value problem**
  - How the transient is affected by the perturbation: different growths of energy, length of the transient, variety of temporal scales;
  - Asymptotic good agreement with modal analysis and with experimental data (in terms of frequency and wavelength);
Conclusions

- Modal analysis
  - Synthetic perturbation hypothesis (saddle point sequence);
  - Absolute instability pockets in the intermediate wake ($Re = 50, 100$);
  - Frequency in good agreement with numerical and experimental data;
  - *No information on the early time history of the perturbation*;

- Initial-value problem
  - How the transient is affected by the perturbation: different growths of energy, length of the transient, variety of temporal scales;
  - Asymptotic good agreement with modal analysis and with experimental data (in terms of frequency and wavelength);
  - Multiscaling $O(1)$ well approximates the full linear problem;
Conclusions

- **Modal analysis**
  - Synthetic perturbation hypothesis (saddle point sequence);
  - Absolute instability pockets in the intermediate wake ($Re = 50, 100$);
  - Frequency in good agreement with numerical and experimental data;
  - *No information on the early time history of the perturbation*;

- **Initial-value problem**
  - How the transient is affected by the perturbation: different growths of energy, length of the transient, variety of temporal scales;
  - Asymptotic good agreement with modal analysis and with experimental data (in terms of frequency and wavelength);
  - Multiscaling $O(1)$ well approximates the full linear problem;
  - *Rich description of the transient but more difficult handling of the parameters.*
Next Steps

- Energy spectrum of a general pre-unstable large set of *multiple transient three-dimensional waves* (accepted for *EFMC8, 2010*).
Next Steps

- Energy spectrum of a general pre-unstable large set of *multiple transient three-dimensional waves* (accepted for *EFMC8*, 2010).

  ⇒ Comparison with the Kolmogorov’s 5/3 law;
Next Steps

- Energy spectrum of a general pre-unstable large set of *multiple transient three-dimensional waves* (accepted for EFMC8, 2010).
  ⇒ Comparison with the Kolmogorov’s 5/3 law;
- Initial-value problem for the cross flow boundary layer ($U(y), W(y)$);
- Energy spectrum of a general pre-unstable large set of *multiple transient three-dimensional waves* (accepted for EFMC8, 2010).
  ⇒ Comparison with the Kolmogorov’s 5/3 law;
- Initial-value problem for the cross flow boundary layer \((U(y), W(y))\);
Next Steps

- Energy spectrum of a general pre-unstable large set of multiple transient three-dimensional waves (accepted for EFMC8, 2010).
  \[ \Rightarrow \text{Comparison with the Kolmogorov's 5/3 law;} \]
- Initial-value problem for the cross flow boundary layer \((U(y), W(y))\);
  \[ \Rightarrow \text{Analytical solution of multiscaling O(1).} \]