Spatial pattern formation induced by stochastic processes

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- Deterministic models have been studied for quite a long time (*Turing 1952*, *Cross & Hohenberg 1993*) with a number of applications to environmental processes (*Borgogno et al. 2009*, *von Hardenberg et al. 2010*, *Manor & Shnerb 2008*, *Couteron & Lejeune 2001*, *Rietkerk & Van de Koppel 2008*, *Kefi et al. 2007*, *Lefever et al. 2009*).

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- Since these models use complicated non-linear terms for the local dynamics and the multiplicative noise terms, their process-based interpretation is often not straightforward.

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  - Rich literature (unlike Gaussian colored or dichotomous noise).
- We call patterned a field that exhibits an ordered state with organized spatial structures. This definition is often adopted in the environmental sciences, where the concomitance of many processes can prevent the organization of the system with a clear dominant wavelength.

Stochastic modeling: general framework

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- $h(\phi)F(t)$: time-dependent forcing term, which can be modulated by a function, $h(\phi)$, of the local state of the system $\Rightarrow$ seasonal phenomena (*phreatic aquifer*).

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  - Structure function (SF): prognostic tool able to assess the presence of a selected wavelength in the spatial field;

Scarsoglio, Laio, D'Odorico, Ridolfi, *Math. BioSci.*, 2011.

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- Numerical simulations:
  - Heun's predictor corrector scheme, 2D square lattice with 128x128 sites;
  - periodic BCs, ICs given by uniformly distributed random numbers between [-0.01, 0.01].

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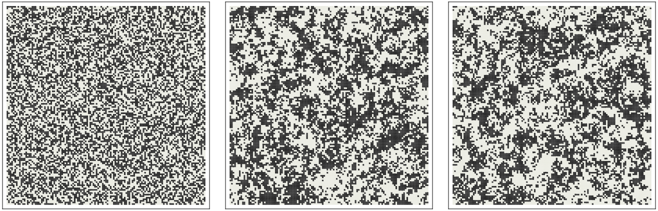
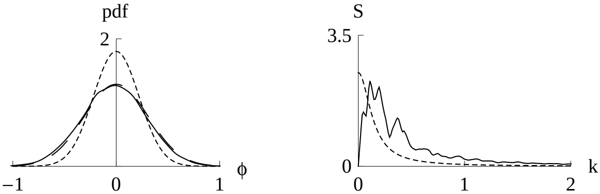
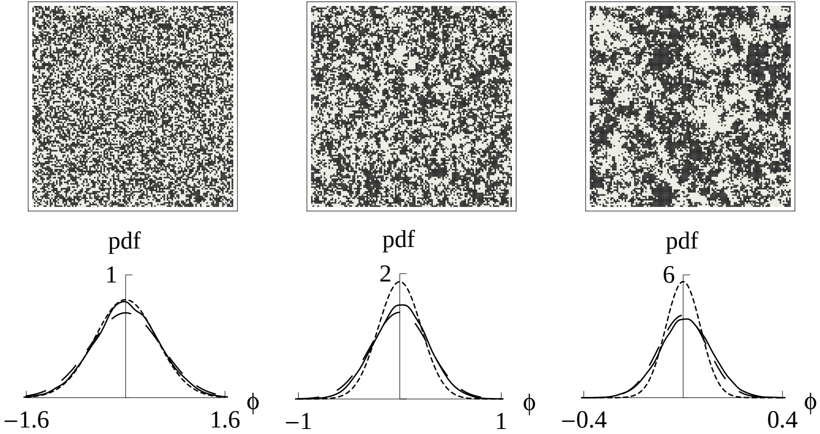
Steady and multiscale patterns





(top) Numerical simulation of ϕ at $t = 0, 10, 100$, $D = 12.5$, $s = 2.5$. (bottom) Pdf (solid: numerical simulation, dotted: classic MFA, dashed: corrected MFA) and azimuthal-averaged power spectrum S (solid: numerical simulation, dotted: SF) of ϕ at $t = 100$.

Role of D



(top) Numerical simulation of ϕ at $t = 100$, $s = 0.5$, $D = 0.25, 2.5, 25$ (left to right). (bottom) Pdf of ϕ (solid: numerical simulation, dotted: classic MFA, dashed: corrected MFA).

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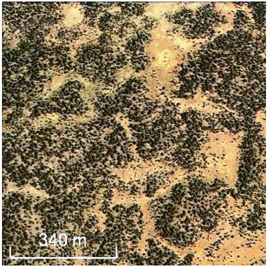

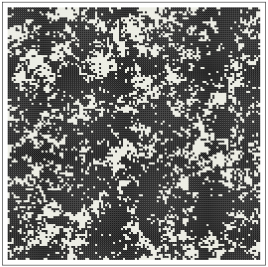

(left) Aerial photograph of vegetation pattern in New Mexico (34°47'N, 108°21'O) and (right) numerical simulation at $t = 100$, $D = 20$, $s = 1$, $\mu = 0.1$.

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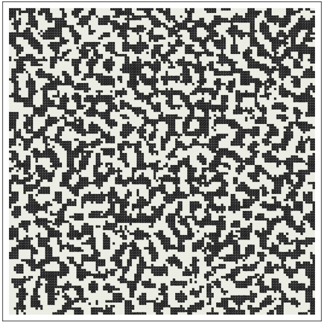

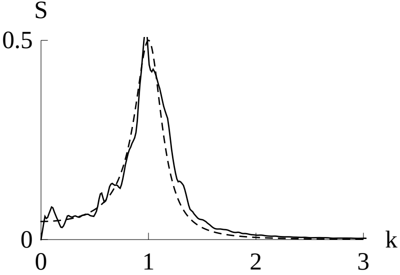

(left) Numerical simulation of ϕ at $t = 100$, $s = 0.5$, $D = 10$, $k_0 = 1$. (right) Azimuthal-averaged power spectrum S (solid: numerical simulation, dotted: SF).

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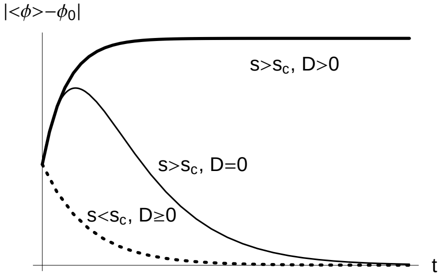

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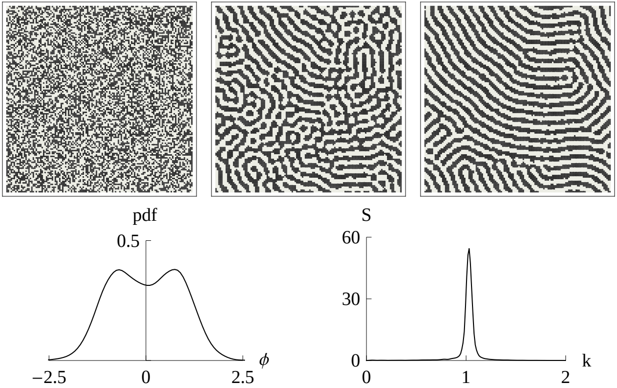

(top) Numerical simulation of the spatial field ϕ at $t = 0, 10, 100$, with $D = 15$, $s = 2.5$, $k_0 = 1$. (bottom) Pdf and azimuthal-averaged power spectrum S at $t = 100$.

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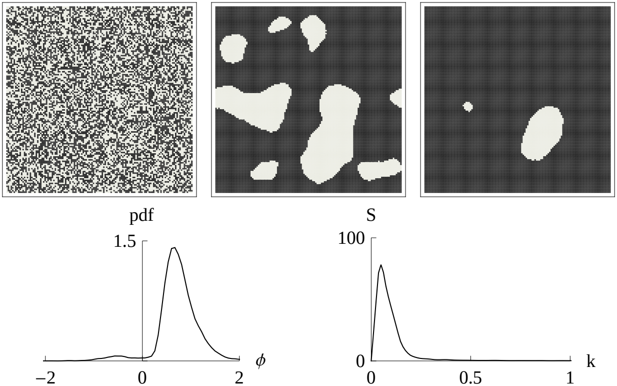

(top) Numerical simulation of the spatial field ϕ at $t = 0, 10, 40$, with $D = 5$, $s = 2$.
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- A suitable synchronization between the frequency of the random transitions and the frequency of the periodic forcing creates a sort of *resonance* $\Rightarrow$ regular transitions between the two stable states (*Gammaitoni et al. 1998*, *Wellens et al. 2004*, *Lindner et al. 1995*).

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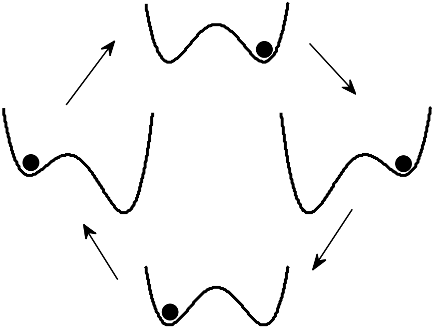

Spatio-Temporal Dynamics

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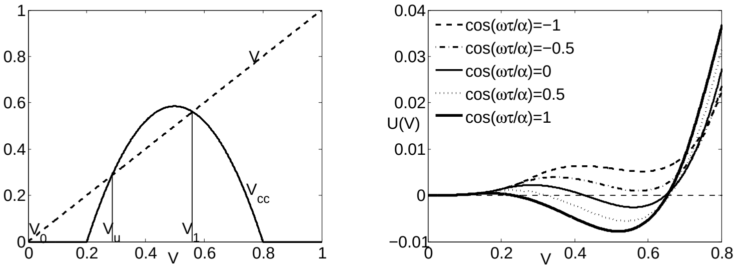
- A suitable cooperation between the three terms is able to give rise to ordered structures which show spatial and temporal coherence, and which are statistically steady in time.

Spatio-temporal stochastic model

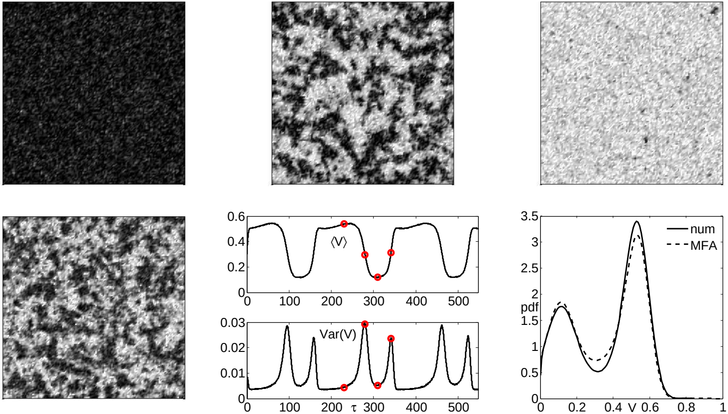
$$\frac{\partial V}{\partial \tau} = V(V_{cc} - V) + \xi(\mathbf{r}, \tau) + D\nabla^2 V$$

- double-well potential $U(V)$, with $\frac{dV}{d\tau} = -\frac{dU}{dV}$



(left) Ecosystem carrying capacity, V_{cc}, and (right) potential, $U(V)$.

Results



Numerical simulation: $s = 0.012$, $D = 0.2$, $A = 0.08$, $\alpha = 0.5$/d, $\beta = 1$, $a = 13$, $d_{sup} = 1.8$, $d_{inf} = 1.2$, $T = 365$ days.

Conclusions

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  - A spatial coupling term which provides spatial coherence.
- For high enough multiplicative noise intensity, the spatial coupling exploits the initial instability giving rise to ordered structures;
- Since noisy fluctuations are always present in real systems and pattern formation, here described, is completely noise-induced, randomness can actually promote spatial coherence in different environmental processes.

Bounded Noise

$$\frac{\partial \phi}{\partial t} = -\phi + D\nabla^2\phi + \xi$$

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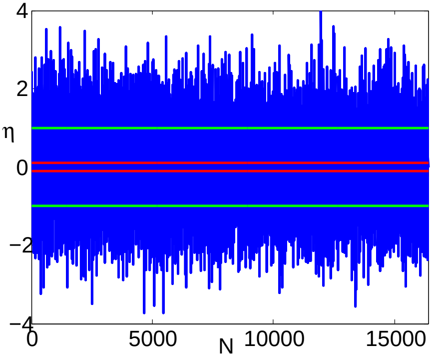
- $D = 50$, $\xi = \sqrt{2s\delta t}\eta$, $s = 3$.



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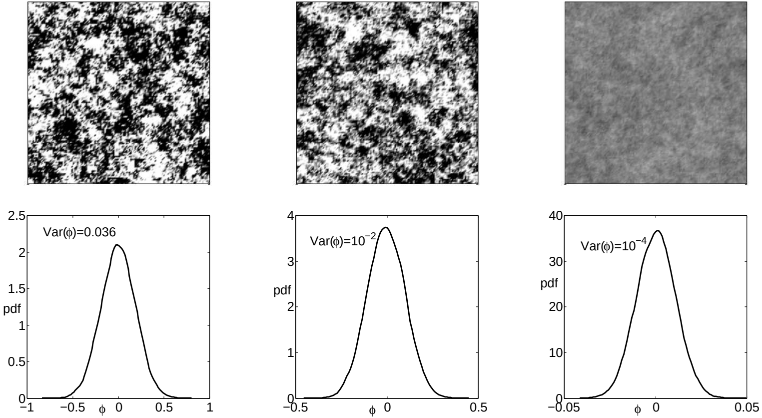

(left): η unbounded. (middle): $\eta \in [-1, 1]$. (right): $\eta \in [-0.1, 0.1]$.