Spatial pattern formation induced by stochastic processes

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June 22nd, 2011
Spatial patterns

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  ⇒ hydrodynamic systems (e.g. Rayleigh-Bénard convection), plant ecosystems (e.g. dryland and riparian vegetation), biochemical and neural systems, etc;
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- Since these models use complicated non-linear terms for the local dynamics and the multiplicative noise terms, their process-based interpretation is often not straightforward.
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We call patterned a field that exhibits an ordered state with organized spatial structures. This definition is often adopted in the environmental sciences, where the concomitance of many processes can prevent the organization of the system with a clear dominant wavelength.
Stochastic modeling: general framework

Temporal evolution of the state variable $\phi$ at any point $r = (x, y)$:

$$\frac{\partial \phi}{\partial t} = f(\phi) + g(\phi)\xi(r, t) + DL[\phi] + h(\phi)F(t)$$
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- \( h(\phi)F(t) \): time-dependent forcing term, which can be modulated by a function, \( h(\phi) \), of the local state of the system \( \Rightarrow \) seasonal phenomena (phreatic aquifer).
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- Additive noise does not play the role of a precursor of a phase transition in a deterministic system close to a bifurcation point, since there is no bifurcation in the deterministic dynamics;
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- Analytical tools:
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- Numerical simulations:
  - Heun’s predictor corrector scheme, 2D square lattice with 128x128 sites;
  - periodic BCs, ICs given by uniformly distributed random numbers between \([-0.01, 0.01]\).

Steady and multiscale patterns

(top) Numerical simulation of $\phi$ at $t = 0, 10, 100$, $D = 12.5$, $s = 2.5$. (bottom) Pdf (solid: numerical simulation, dotted: classic MFA, dashed: corrected MFA) and azimuthal-averaged power spectrum $S$ (solid: numerical simulation, dotted: SF) of $\phi$ at $t = 100$. 
Role of $D$

(top) Numerical simulation of $\phi$ at $t = 100$, $s = 0.5$, $D = 0.25, 2.5, 25$ (left to right).

(bottom) Pdf of $\phi$ (solid: numerical simulation, dotted: classic MFA, dashed: corrected MFA).
Comparison with vegetation pattern

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(left) Aerial photograph of vegetation pattern in New Mexico \((34^\circ 47'N, 108^\circ 21'O)\) and (right) numerical simulation at \(t = 100, D = 20, s = 1, \mu = 0.1\).
Steady and periodic patterns

\[
\frac{\partial \phi}{\partial t} = -\phi - D(\nabla^2 + k_0^2 \phi^2) + \xi
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(left) Numerical simulation of \( \phi \) at \( t = 100, \ s = 0.5, \ D = 10, \ k_0 = 1 \). (right) Azimuthal-averaged power spectrum \( S \) (solid: numerical simulation, dotted: SF).

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Short-term instability and spatial coupling

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- The spatial coupling exploits this initial instability, giving rise to the pattern and stabilizing it.

For $s < s_c$, the system remains blocked in the disordered phase and no patterns occur. Only transiently, the spatial coupling might be able to induce patterns that fade away at steady state;

For $s > s_c$, the spatial term can take advantage from the noise-induced short-term instability and prevents the decay to zero. The spatial coupling traps the system in a new ordered state.

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(top) Numerical simulation of the spatial field \( \phi \) at \( t = 0, 10, 100 \), with \( D = 15 \), \( s = 2.5 \), \( k_0 = 1 \). (bottom) Pdf and azimuthal-averaged power spectrum \( S \) at \( t = 100 \).
Transient and multiscale patterns

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(top) Numerical simulation of the spatial field \( \phi \) at \( t = 0, 10, 40 \), with \( D = 5, s = 2 \).
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Temporal Dynamics

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- A suitable synchronization between the frequency of the random transitions and the frequency of the periodic forcing creates a sort of **resonance** ⇒ regular transitions between the two stable states (Gammaitoni et al. 1998, Wellens et al. 2004, Lindner et al. 1995).
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Spatio-Temporal Dynamics

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A suitable cooperation between the three terms is able to give rise to ordered structures which show spatial and temporal coherence, and which are statistically steady in time.
Spatio-temporal stochastic model

\[ \frac{\partial V}{\partial \tau} = V(V_{cc} - V) + \xi(r, \tau) + D\nabla^2 V \]

- double-well potential \( U(V) \), with \( \frac{dV}{d\tau} = -\frac{dU}{dV} \)

(Left) Ecosystem carrying capacity, \( V_{cc} \), and (right) potential, \( U(V) \).
Numerical simulation: $s = 0.012, D = 0.2, A = 0.08, \alpha = 0.5/d, \beta = 1, a = 13, d_{sup} = 1.8, d_{inf} = 1.2, T = 365$ days.
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- A spatial coupling term which provides spatial coherence.
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  - An additive noise able to maintain the dynamics away from the uniform steady state;
  - A spatial coupling term which provides spatial coherence.
- For high enough multiplicative noise intensity, the spatial coupling exploits the initial instability giving rise to ordered structures;

Since noisy fluctuations are always present in real systems and pattern formation, here described, is completely noise-induced, randomness can actually promote spatial coherence in different environmental processes.
Conclusions

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Bounded Noise

\[ \frac{\partial \phi}{\partial t} = -\phi + D \nabla^2 \phi + \xi \]
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- \( D = 50, \xi = \sqrt{2s\delta t} \eta, s = 3. \)
Bounded Noise

(left): $\eta$ unbounded. (middle): $\eta \in [-1, 1]$. (right): $\eta \in [-0.1, 0.1]$. 

$\text{Var}(\phi) = 0.036$

$\text{Var}(\phi) = 10^{-2}$

$\text{Var}(\phi) = 10^{-4}$