Noise-induced spatial pattern formation in dynamical systems

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  ⇒ **An increase of the noise can produce a more regular behaviour** (*counterintuitive!*).

Spatiotemporal Dynamics

Stochastic differential equations

Temporal evolution of the state variable ϕ at any point $\mathbf{r} = (x, y)$:

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- $D\mathcal{L}[\phi]$: spatial coupling. $\mathcal{L}$ represents the Laplacian (∇^2) or the Swift-Hohemberg $(\nabla^2+k_0^2)^2$ coupling (k_0 is the selected wavenumber). D is the strength of the spatial coupling.

Stochastic Model: Steady and Periodic Patterns

$$\frac{\partial \phi}{\partial t} = a\phi + \xi - D(\nabla^2 + k_0^2)^2 \phi$$

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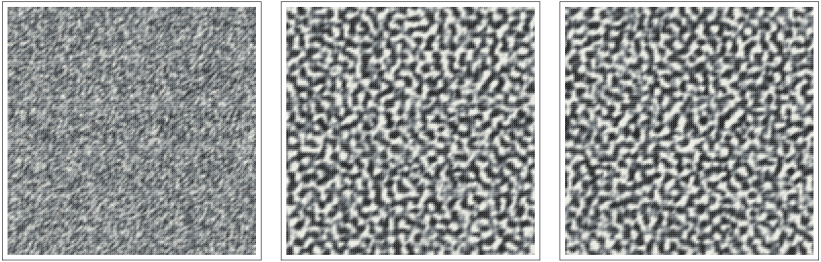

$t = 0, 50, 100$. $a = -1$, $D = 10$, $s = 1$, $k_0 = 1$, 128x128 pixels, periodic BCs.

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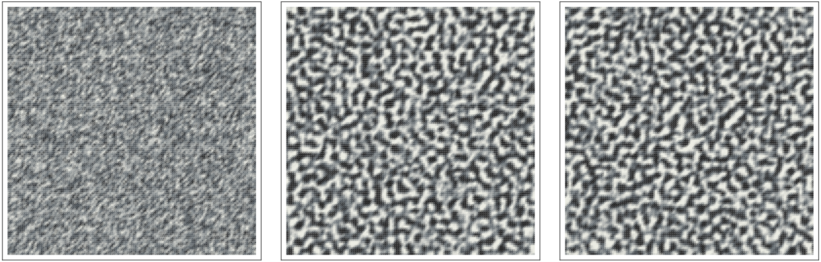

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- The deterministic dynamics ($\xi = 0$) show transient patterns.

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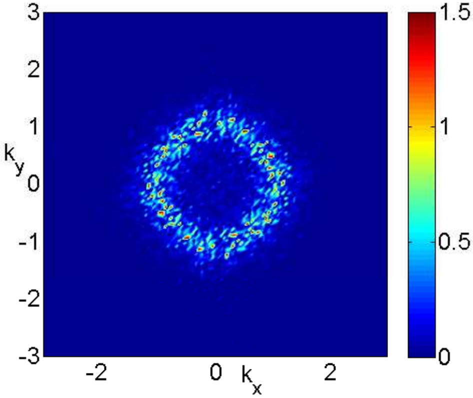

Two-dimensional power spectrum (logarithmic scale), $t = 100$.

Stochastic Model: Steady and Multiscale Patterns

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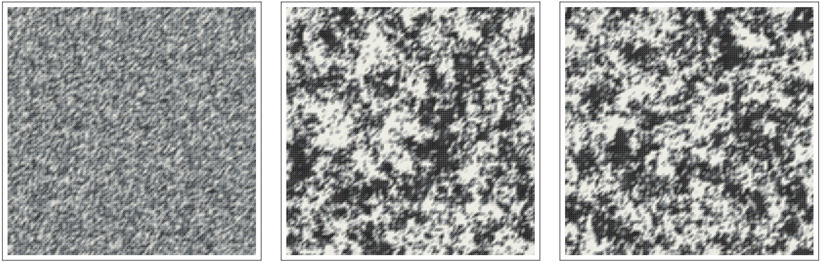

$t = 0, 200, 400$. $a = -0.1$, $D = 10$, $s = 10$, 128x128 pixels, periodic BCs.

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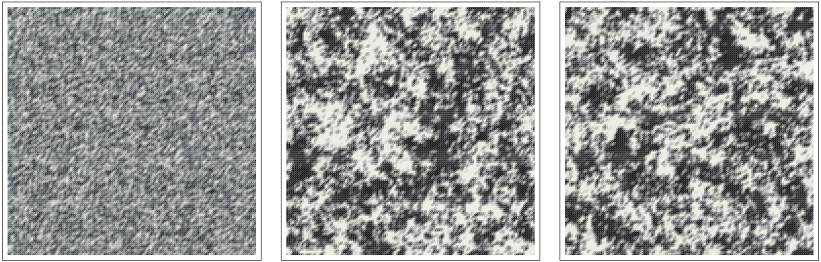

$t = 0, 200, 400$. $a = -0.1$, $D = 10$, $s = 10$, 128x128 pixels, periodic BCs.

- The deterministic dynamics ($\xi = 0$) are unable to generate any type of pattern.

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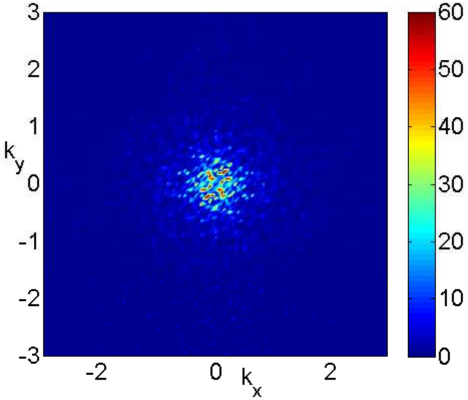

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Stochastic Model: Comparison with Vegetation Pattern

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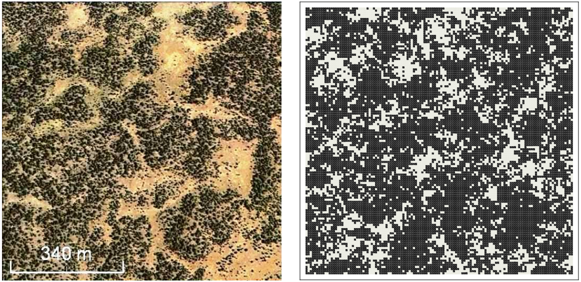

(left) Aerial photograph of vegetation pattern in New Mexico (34°47'N, 108°21'O) and (right) numerical simulation at $t = 100$, $a = -1$, $D = 80$, $s = 2$, $\mu = 0.1$. Google Earth imagery © Google Inc. Used with permission.

Scarsoglio, Laio, Ridolfi & D'Odorico, submitted to *Phys. Rev. Lett.*

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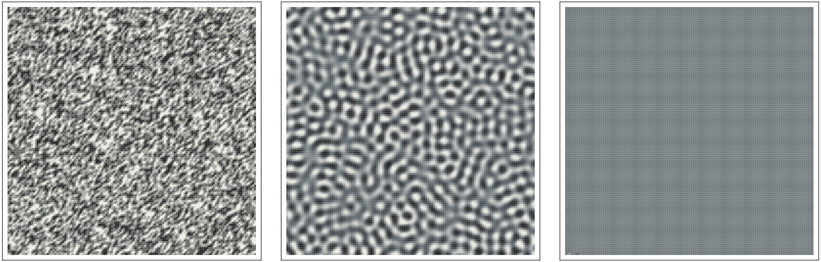

$t = 0, 1, 10$. $D = 15$, $s = 1$, 128x128 pixels, periodic BCs.

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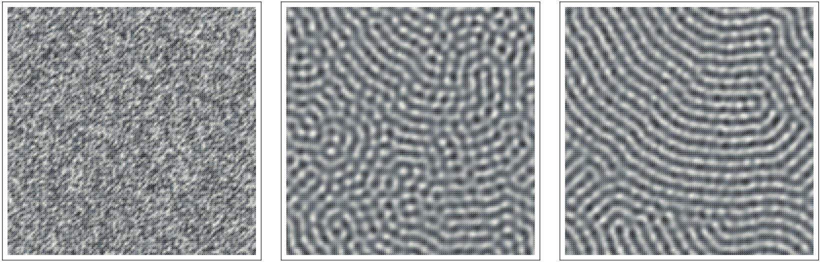

$t = 0, 10, 100$. $D = 15$, $s = 5$, 128x128 pixels, periodic BCs.

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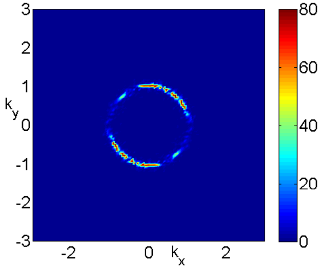

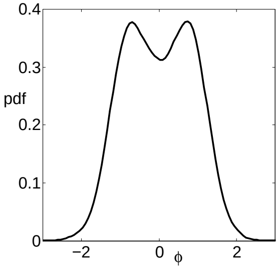

(left) Two-dimensional power spectrum and (right) pdf, $t = 100$. $D = 15$, $s = 5$.

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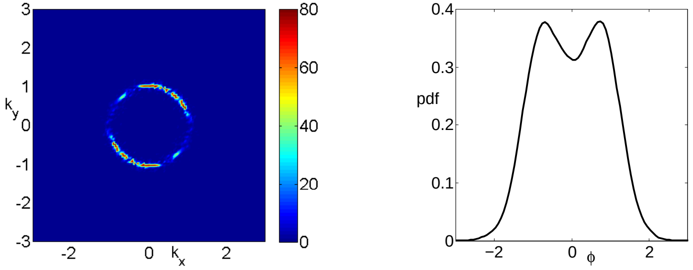

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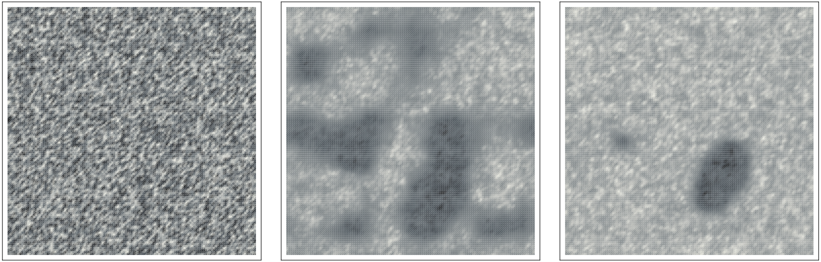

$t = 0, 10, 40$. $D = 20$, $s = 4$, 128x128 pixels, periodic BCs.

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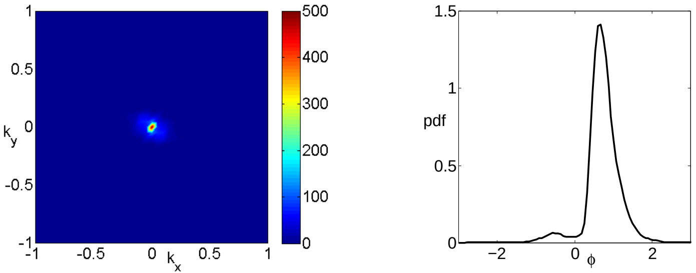

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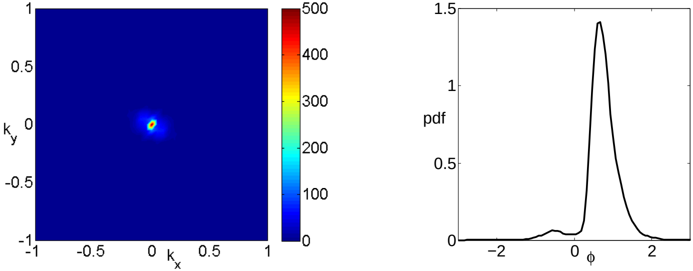

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