Linear generation of multiple time scales by three-dimensional unstable perturbations

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Introduction

- Appearance of different time scales during the transient growth of a small two-dimensional or three-dimensional perturbation applied to the cylinder wake:
  - Temporal scales different each other and different from the values obtained in the asymptotic limit;
  - Variety of time scales associated to a given specific value of the instability wavelength;
  - Universal behaviour revealed by asymmetric oblique or longitudinal disturbances;

- These phenomena are developing in the context of the linear dynamics;

- The frequency determination is validated through the asymptotic comparison with normal mode theory and experimental results.
Base flow: 2D cylinder wake

- Flow behind a circular cylinder steady, incompressible and viscous;
- Approximation of 2D asymptotic Navier-Stokes expansions (Belen & Tordella, 2003) $U(y; x_0, \text{Re})$
The initial-value problem formulation

- Linear, three-dimensional perturbative equations in terms of velocity and vorticity (Criminale & Drazin, 1990);
- Laplace-Fourier transform of perturbation quantities in \(x\) and \(z\) directions, \(\alpha\) complex, \(\gamma\) real (Scarsoglio et al., 2009);

\[\alpha_r = \text{longitudinal wavenumber}\]

\[\gamma = \text{transversal wavenumber}\]

\[\Phi = \text{angle of obliquity}\]

\[k = \text{polar wavenumber}\]

\[\alpha_i = \text{spatial damping rate}\]
The governing partial differential equations are:

\[
\begin{align*}
\frac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i) \hat{v} &= \tilde{\Gamma} \\
\frac{\partial \tilde{\Gamma}}{\partial t} &= (i\alpha_r - \alpha_i) \left( \frac{d^2 U}{dy^2} \hat{v} - U \tilde{\Gamma} \right) + \frac{1}{Re} \left[ \frac{\partial^2 \tilde{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i) \tilde{\Gamma} \right] \\
\frac{\partial \tilde{\omega}_y}{\partial t} &= -(i\alpha_r - \alpha_i) U \tilde{\omega}_y - i\gamma \frac{dU}{dy} \hat{v} + \frac{1}{Re} \left[ \frac{\partial^2 \tilde{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i) \tilde{\omega}_y \right]
\end{align*}
\]

where the transversal velocity and vorticity components are indicated as \( \hat{v} \) and \( \tilde{\omega}_y \) respectively, while \( \tilde{\Gamma} \) is defined through the kinematic relation \( \tilde{\Gamma} = \partial_x \tilde{\omega}_z - \partial_z \tilde{\omega}_x \).

**Periodic initial conditions:**

\[
\begin{align*}
\tilde{\omega}_y(0, y) &= 0 \\
\hat{v}(0, y) &= e^{-y^2} \sin(y) \quad \text{asymmetric} \\
\text{or} \quad &\quad \hat{v}(0, y) = e^{-y^2} \cos(y) \quad \text{symmetric}
\end{align*}
\]

**The perturbation velocity field has to vanish in the free stream.**
Early transient and asymptotic behaviour

- Kinetic energy density $e$ of the perturbation:

$$e(t; \alpha, \gamma, Re) = \frac{1}{2y_d} \int_{-y_d}^{+y_d} \left( |\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2 \right) dy$$

$$= \frac{1}{2y_d} \frac{1}{|\alpha^2 + \gamma^2|} \int_{-y_d}^{+y_d} \left( |\frac{\partial \hat{v}}{\partial y}|^2 + |\alpha^2 + \gamma^2||\hat{v}|^2 + |\hat{w_y}|^2 \right) dy$$

- The amplification factor $G$:

$$G(t; \alpha, \gamma) = \frac{e(t; \alpha, \gamma)}{e(t = 0; \alpha, \gamma)}$$

measures the growth of the perturbation at time $t$. 
- The temporal growth rate $r$ (Lasseigne et al., 1999) is

$$ r(t; \alpha, \gamma) = \frac{\log|e(t; \alpha, \gamma)|}{2t}, \quad t > 0 $$

- The angular frequency (pulsation) $\omega$ (Whitham, 1974) is

$$ \omega(t) = \frac{d\varphi(t)}{dt} \quad \varphi \text{ time phase} $$

$$ \bar{v}(y, t; \alpha, \gamma, Re) = A_t(y; \alpha, \gamma, Re)e^{i\varphi(t)} $$
Results

(a)-(b) Effect of the symmetry of the perturbation. The amplification factor $G$. (a) asymmetric initial condition, (b) symmetric initial condition. Intermediate ($x_0=10$, solid curves) and far field ($x_0=50$, dashed curves) wake configurations.
Pulsation $\omega$. Intermediate wake configuration ($x_0=10$), symmetric and asymmetric cases.
(a)-(b) Effect of the angle of obliquity $\Phi$. (a) The amplification factor $G$ and (b) the temporal growth rate $r$ as functions of time. Asymmetric initial condition, $\Phi = 0$ (solid curves), $\Phi = \pi/2$ (dashed curves).
(a, b, c) pulsation $\omega$ and (d, e, f) temporal growth rate $r$ for present results (asymmetric case: triangles, symmetric case: circles), modal analysis (Tordella et al. 2006) (solid curves) and experimental data (Williamson 1989) (squares). $\alpha_i=0.05$, $\Phi=0$, $x_0=10$, Re=50, 70, 100.
Conclusions

- Variety of temporal scales shown by the transient in the context of the linear dynamics;

- For asymmetric oblique or longitudinal waves it is possible to count up to five different time scales:
  (i) The temporal scale $D/U \sim 1$ related to the base flow;
  (ii) The length of the transient (200-300 time units);
  (iii)-(iv) The scales associated to the instability frequency in the early transient (about 25 time units) and in the asymptotic state (about 15 time units);
  (v) The modulation of the pulsation in the early transient (about 35-40 time units);

- Asymptotically good agreement with normal mode theory and experimental data.