Power-law decay of the energy spectrum in linearized perturbed systems

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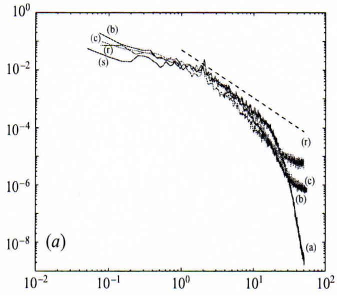

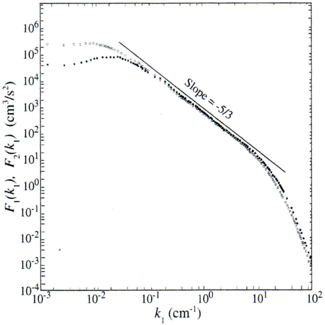

(left) *Evangelinos & Karniadakis, JFM 1999*. (right) *Champagne, JFM 1978*.

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- The perturbative evolution is ruled by the **initial-value problem** associated to the Navier-Stokes linearized formulation.

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- We determine the **energy decay exponent of arbitrary longitudinal and transversal perturbations in their asymptotic states** and we compare it with the -5/3 Kolmogorov decay.

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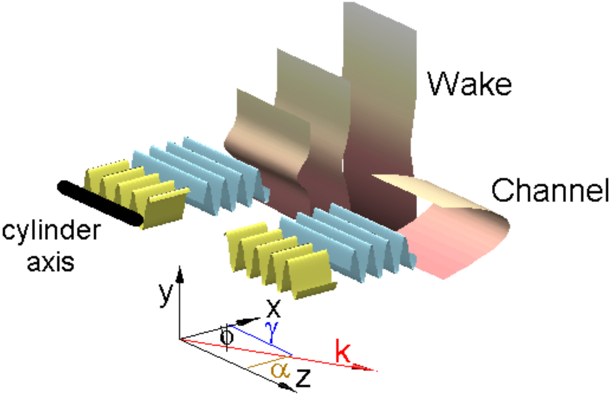

Perturbative equations

- Perturbative linearized system:

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\frac{\partial \hat{\Gamma}}{\partial t} &= (i\alpha_r - \alpha_i)(\frac{d^2U}{dy^2}\hat{v} - U\hat{\Gamma}) + \frac{1}{Re}[\frac{\partial^2\hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\Gamma}] \\
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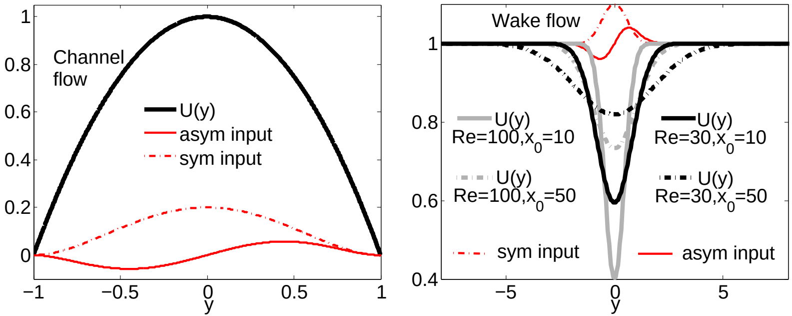

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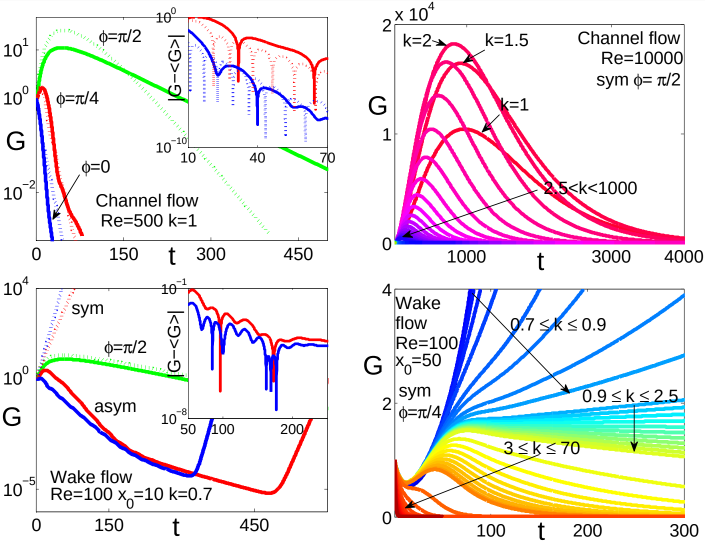
- Angular frequency (pulsation) ω (Whitham, 1974):

$$\omega(t;\alpha,\gamma)=\frac{d\varphi(t)}{dt}, \qquad \varphi \;\; \text{time phase}$$

Relevant transient behaviours



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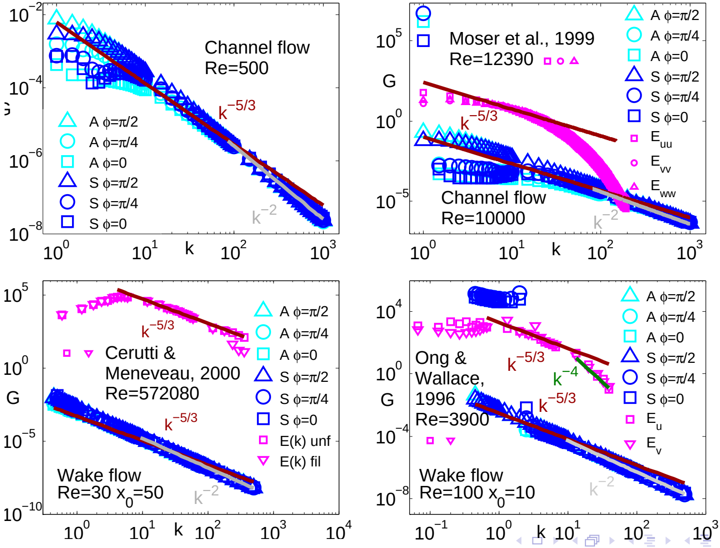
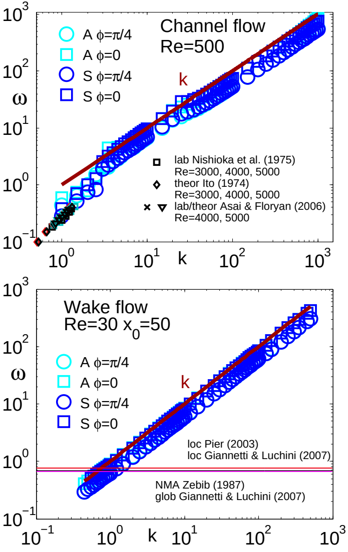
- The energy spectrum is evaluated as the wavenumber distribution of the perturbation kinetic energy density, G(k);
- **The spectral representation is determined by comparing the energy of the waves when they are exiting their transient state**;
- Every perturbation has its characteristic transient exiting time, T_e;
- The asymptotic condition is reached when the perturbative wave exceeds the transient exiting time, T_e, that is when $r \sim const$ is satisfied for stable and unstable waves.

Scarsoglio, De Santi & Tordella, ETC XIII, 2011.

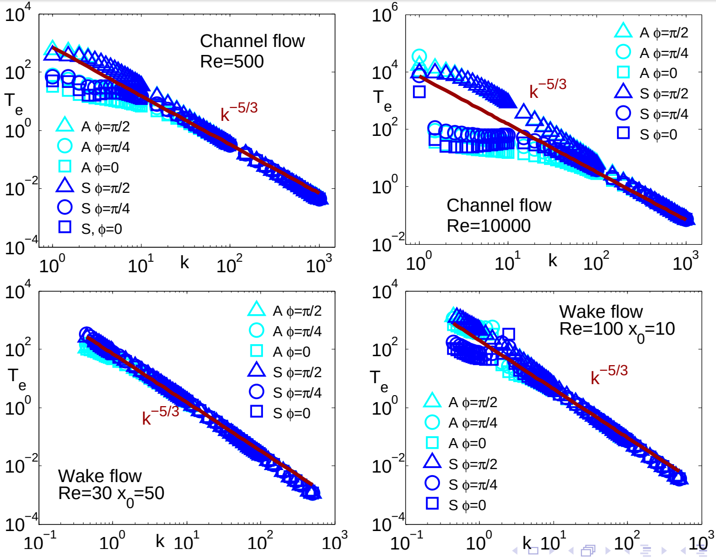
Energy $G(k)$ at the asymptotic state ($r \sim$const)



Pulsation $\omega(k)$ at the asymptotic state ($r \sim$const)



Transient exiting time $T_e(k)$



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- Scale-invariance of G and T_e at different (stable and unstable) Reynolds numbers and for different shear flows;
- **The spectral power-law scaling of inertial waves is a general dynamical property which encompasses the nonlinear interaction**;
- **The** $-5/3$ **power-law scaling in the intermediate range seems to be an intrinsic property of the Navier-Stokes solutions.**