Hydrodynamic linear stability of the two-dimensional bluff-body wake through modal analysis and initial-value problem formulation

Stefania Scarsoglio

Dottorato di Ricerca in Fluidodinamica – XX ciclo

*Dipartimento di Ingegneria Aeronautica e Spaziale*
*Politecnico di Torino*

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Outline

- Physical problem
- Normal mode analysis
- Entrainment evolution
- Initial-value problem
- Multiscale analysis for the stability of long waves
- Conclusions
Physical Problem

- Flow behind a circular cylinder steady, incompressible and viscous;
- Approximation of 2D asymptotic Navier-Stokes expansions (Belan & Tordella, 2003), $20 \leq \text{Re} \leq 100$. 
Normal mode analysis

The linearized perturbative equation in terms of stream function \( \psi(x, y, t) \) is

\[
\frac{\partial}{\partial t} \nabla^2 \psi + (\partial_x \nabla^2 \psi) \psi_y + \psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \psi) \psi_x - \psi_x \partial_y \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi
\]

**Normal mode hypothesis**

\[ \psi(x, y, t) = \varphi(x, y, t) e^{i(h_0 x - \sigma_0 t)} \]

- \( \varphi(x, y, t) \) complex eigenfunction

- \( h_0 = k_0 + i s_0 \) complex wave number
- \( s_0 \): spatial growth rate
- \( \sigma_0 = \omega_0 + i r_0 \) complex frequency
- \( r_0 \): temporal growth rate
- \( k_0 \): wave number
- \( \omega_0 \): frequency

**Convective instability:** \( r_0 < 0 \) for all modes, \( s_0 < 0 \) for at least one mode.

**Absolute instability:** \( r_0 > 0 \), \( v_g = \partial \sigma_0 / \partial h_0 = 0 \) for at least one mode.
Stability analysis through multiscale approach

- Slow variables: \( x_1 = \varepsilon x, \, t_1 = \varepsilon t, \, \varepsilon = 1/\text{Re} \).
- Hypothesis: \( \psi(x, y, t) \) and \( \psi(x, y) \) are expansions in terms of \( \varepsilon \):

\[
(\text{ODE dependent on } \varphi_0) + \varepsilon (\text{ODE dependent on } \varphi_0, \varphi_1) + O(\varepsilon^2)
\]

**Order zero theory** Homogeneous Orr-Sommerfeld equation

\[
\begin{align*}
A \varphi_0 &= \sigma_0 B \varphi_0 \\
\varphi_0 &\to 0, \ |y| \to \infty \\
\partial_1 \varphi_0 &\to 0, \ |y| \to \infty \\
\end{align*}
\]

\( \longrightarrow \) eigenfunctions \( \varphi_0 \) and a discrete set of eigenvalues \( \sigma_{0n} \)

**First order theory** Non homogeneous Orr-Sommerfeld equation

\[
\begin{align*}
A \varphi_1 &= \sigma_0 B \varphi_1 + M \varphi_0 \\
\varphi_1 &\to 0, \ |y| \to \infty \\
\partial_1 \varphi_1 &\to 0, \ |y| \to \infty \\
\end{align*}
\]

\[
M = \left\{ Re\left(2h_0 \varphi_0 - 3h_0^2 u_0 - \partial_y^2 u_0\right) + 4ih_0^3\right\} \partial_{x_1} \\
+ \left(Reu_0 - 4ih_0\right) \partial_{x_1 y}^3 - Re\vartheta_1 (\partial_y^3 - h_0^2 \partial_y) + Re \partial_y^2 v_1 \partial_y \\
+ ih_0 Re \left[ u_1 (\partial_y^2 - h_0^2) - \partial_y^2 u_1 \right] + Re (\partial_y^2 - h_0^2) \partial_{t_1}\right\}
\]
Perturbative hypothesis – Saddle points sequence

- For fixed values of $\times$ and $\text{Re}$ the saddle points $(h_{0s}, \sigma_{0s})$ of the dispersion relation $\sigma_{0}=\sigma_{0}(h_{0}, x, \text{Re})$ satisfy the condition $\partial \sigma_{0}/ \partial h_{0} = 0$;
- The system is perturbed at every station with the most unstable characteristics at order zero.

Re=35, $x/D=4$. Level curves, $\omega_{0}=\text{cost}$ (thick curves), $r_{0}=\text{cost}$ (thin curves).
\( \omega_0(k_0, s_0), r_0(k_0, s_0). \text{Re} = 35, x/D = 4. \)
Frequency. Comparison between present solution (accuracy $\Delta \omega = 0.05$), Zebib's numerical study (1987), Pier's direct numerical simulations (2002), Williamson's experimental results (1988).

Eigenfunctions and eigenvalues asymptotic theory

An asymptotic analysis for the Orr-Sommerfeld zero order problem is proposed. For \( x \to \infty \) the eigenvalue problem becomes

\[
\left\{ \partial_y^2 - h_0^2 - ih_0 Re u_0 \right\} f = -i Re \sigma_0 f
\]

\( f \to 0 \) as \( |y| \to \infty \)

where \( f(x, y) = (\partial_y^2 - h_0^2) \varphi_0(x, y) \)

\( k_0 \sim 0, \text{ as } x \to \infty \)

\( s_0 < 0, \; \forall x \)

\( \omega_0 \sim 0, \; r_0, s_0 \to 0, \text{ as } x \to \infty \)

\( r_0 \sim s_0 + \frac{s_0^2}{Re}, \text{ as } x \to \infty \)
Entrainment evolution

\[ Q(x) = \frac{1}{2z_w \delta} \int_{-z_w}^{z_w} \int_0^\delta U(x, y) \, dy \, dz \]

Volumetric flow rate

\[ E(x) = \frac{dQ(x)}{dx} \]

Entrainment

20 \leq Re \leq 100
Initial-value problem

- Linear, three-dimensional perturbative equations in terms of vorticity and velocity (Criminale & Drazin, 1990);

- Base flow parametric in $x$ and $Re$ $\rightarrow U(y; x_0, Re)$

- Laplace-Fourier transform in $x$ and $z$ directions for perturbation quantities:

$$\begin{align*}
\hat{\tilde{v}}_{yy} &- (k^2 - \alpha_i^2 + 2ik\cos(\phi)\alpha_i)\hat{\tilde{v}} = \hat{\Gamma} \\
\frac{\partial \hat{\tilde{v}}}{\partial t} &= - (ik\cos(\phi) - \alpha_i)U\hat{\tilde{v}} + (ik\cos(\phi) - \alpha_i)d^2U \frac{\partial^2 \tilde{v}}{dy^2} \\
\hat{\omega}_y &= \frac{\partial \tilde{u}}{\partial z} - \frac{\partial \tilde{\omega}}{\partial x} \\
\frac{\partial \hat{\omega}_y}{\partial t} &= - (ik\cos(\phi) - \alpha_i)U\hat{\omega}_y - iksin(\phi)dU \frac{\partial \tilde{v}}{dy} \\
\hat{\Gamma} &= \frac{\partial \hat{\omega}_z}{\partial x} - \frac{\partial \hat{\omega}_x}{\partial z}
\end{align*}$$
\[ a_r = k \cos(\Phi) \text{ wavenumber in x-direction} \quad \gamma = k \sin(\Phi) \text{ wavenumber in z-direction} \]

\[ \Phi = \tan^{-1}(\gamma / a_r) \text{ angle of obliquity} \quad k = (a_r^2 + \gamma^2)^{1/2} \text{ polar wavenumber} \]

\[ a_i \geq 0 \text{ spatial damping rate} \]
• Periodic initial conditions for \( \hat{v} = \frac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2ik\cos(\phi)\alpha_i)\hat{v} \)

\[
\begin{cases}
\hat{v}(y, t = 0) = e^{-(y-y_0)^2} \cos(n_0(y - y_0)) & \text{symmetric} \\
\hat{v}(y, t = 0) = e^{-(y-y_0)^2} \sin(n_0(y - y_0)) & \text{asymmetric}
\end{cases}
\]

and \( \hat{\omega}_y(y, t = 0) = 0 \)

• Velocity field vanishing in the free stream.
Early transient and asymptotic behaviour of perturbations

- The growth function $G$ is the normalized kinetic energy density

\[ G(t; \alpha, \gamma) = \frac{e(t; \alpha, \gamma)}{e(t = 0; \alpha, \gamma)} \]

and measures the growth of the perturbation energy at time $t$.

- The temporal growth rate $r$ (Lasseigne et al., 1999) and the angular frequency $\omega$ (Whitham, 1974)

\[ r(t; \alpha, \gamma) = \frac{\log|e(t; \alpha, \gamma)|}{2t}, \quad t > 0 \]

\[ \omega(t; \alpha, \gamma) = \frac{|d\varphi(t; \alpha, \gamma)|}{dt} \]

$\varphi$ perturbation phase
Exploratory analysis of the transient dynamics

(a): Wave spatial evolution in the x direction for \( k = \alpha_r = 0.5, \alpha_i = 0, 0.01, 0.05, 0.1 \).

(b): \( R = 50, y_0 = 0, x_0 = 7, k = 0.5, \phi = 0 \) asymmetric, \( n_0 = 1, \alpha_i = 0, 0.01, 0.05, 0.1 \).
(d): $R=100$, $y_0=0$, $x_0=11.50$, $k=0.7$, asymmetric, $\alpha_i=0.02$, $n_0=1$, $\Phi=0$, $\pi/2$.

(e): $R=100$, $x_0=12$, $k=1.2$, $\alpha_i=0.01$, symmetric, $n_0=1$, $\Phi=\pi/2$, $y_0=0,2,4,6$.

(f): $R=50$, $x_0=14$, $k=0.9$, $\alpha_i=0.15$, asymmetric, $y_0=0$, $\Phi=\pi/2$, $n_0=1,3,5,7$. 
(a)-(b)-(c)-(d): $R=100, y_0=0, k=0.6, \alpha_i=0.02, n_0=1, \Phi=\pi/4, x_0=11$ and 50, symmetric and asymmetric.
Asymptotic fate and comparison with modal analysis

- Asymptotic state: the temporal growth rate \( r \) asymptotes to a constant value \( (dr/dt < \varepsilon \sim 10^{-4}) \).

(a)-(b): \( \text{Re}=50, \; \alpha_i=0.05, \; \phi=0, \; x_0=11, \; n_0=1, \; y_0=0 \). Initial-value problem (triangles: symmetric, circles: asymmetric), normal mode analysis (black curves), experimental data (Williamson 1989, red symbols).
Multiscale analysis for the stability of long waves

- Different scales in the stability analysis:
  - Slow scales (base flow evolution);
  - Fast scales (disturbance dynamics);

- In some flow configurations, long waves can be destabilizing (for example Blasius boundary layer and 3D cross flow boundary layer);

- In such instances the perturbation wavenumber of the unstable wave is much less than $O(1)$.

Small parameter is the polar wavenumber of the perturbation:

$$k \ll 1$$
Full linear system

\[
\begin{align*}
\frac{\partial^2 \hat{v}}{\partial y^2} & \quad - \quad (k^2 - \alpha_i^2 + 2ik\cos(\phi)\alpha_i)\hat{v} = \hat{\Gamma} \\
\frac{\partial \hat{\Gamma}}{\partial t} & \quad = \quad G\hat{\Gamma} + H\hat{v} + K\hat{\omega}_y \\
\frac{\partial \hat{\omega}_y}{\partial t} & \quad = \quad L\hat{\omega}_y + M\hat{v}
\end{align*}
\]

\[ G = G(y; k, \phi, \alpha_i, Re) \]

base flow

\[ (U(x,y;Re), V(x,y;Re)) \]

Multiple scales hypothesis

- Regular perturbation scheme, \( k \ll 1 \):

\[
\begin{align*}
\hat{v} & \quad = \quad \hat{v}_0 + k\hat{v}_1 + k^2\hat{v}_2 + \ldots \\
\hat{\Gamma} & \quad = \quad \hat{\Gamma}_0 + k\hat{\Gamma}_1 + k^2\hat{\Gamma}_2 + \ldots \\
\hat{\omega}_y & \quad = \quad \hat{\omega}_{y0} + k\hat{\omega}_{y1} + k^2\hat{\omega}_{y2} + \ldots
\end{align*}
\]

- Temporal scales: \( t, \tau = kt, T = k^2t \);  
- Spatial scales: \( y, Y = ky \);
Order $O(1)$

\[
\frac{\partial^2 \hat{v}_0}{\partial y^2} + \alpha_i^2 \hat{v}_0 = \hat{\Gamma}_0
\]

\[
\frac{\partial \hat{\Gamma}_0}{\partial t} - G_h \hat{\Gamma}_0 - H_h \hat{v}_0 = 0 \quad G_h = G_h(y; \phi, \alpha_i, Re)
\]

\[
\frac{\partial \hat{\omega}_y}{\partial t} - L_h \hat{\omega}_y = 0
\]

Order $O(k)$

\[
\frac{\partial^2 \hat{v}_1}{\partial y^2} + \alpha_i^2 \hat{v}_1 = -2 \frac{\partial^2 \hat{v}_0}{\partial y \partial Y} + 2i \cos(\phi) \alpha_i \hat{v}_0 + \hat{\Gamma}_1
\]

\[
\frac{\partial \hat{\Gamma}_1}{\partial t} - G_h \hat{\Gamma}_1 - H_h \hat{v}_1 = -\frac{\partial \hat{\Gamma}_0}{\partial \tau} + G_{h-1} \hat{\Gamma}_0 + H_{h-1} \hat{v}_0 + K_{h-1} \hat{\omega}_y
\]

\[
\frac{\partial \hat{\omega}_y}{\partial t} - L_h \hat{\omega}_y = -\frac{\partial \hat{\omega}_y}{\partial \tau} + L_{h-1} \hat{\omega}_y + M_{h-1} \hat{v}_0
\]

\[
G_{h-1} = G_{h-1}(y, Y; \phi, \alpha_i, Re)
\]
Comparison with the full linear problem

(a)-(b): $Re=100, \, k=0.01, \, \Phi=\pi/4, \, x_0=10, \, n_0=1, \, y_0=0$. Full linear problem (black circles: symmetric, black triangles: asymmetric), multiscale $O(1)$ (red circles: symmetric, red triangles: asymmetric).
(a): $R=50$, $y_0=0$, $k=0.03$, $n_0=1$, $x_0=12$, $\Phi=\pi/4$, asymmetric, $\alpha_i=0.04, 0.4$.

(b): $R=100$, $y_0=0$, $n_0=1$, $x_0=27$, $\Phi=0$, symmetric, $\alpha_i=0.2$, $k=0.1, 0.01, 0.001$.

(c): $R=100$, $y_0=0$, $k=0.02$, $x_0=13.50$, $n_0=1$, $\Phi=\pi/2$, $\alpha_i=0.08$, sym and asym.
Conclusions

- Synthetic perturbation hypothesis (saddle point sequence);
- Absolute instability pockets ($\text{Re}=50,100$) found in the intermediate wake;
- Good agreement, in terms of frequency, with numerical and experimental data;
- **No information on the early time history of the perturbation**;
- Different transient growths of energy;
- Asymptotic good agreement with modal analysis and with experimental data (in terms of frequency and wavelength);
- Multiscaling $O(1)$ for long waves well approximates full linear problem.