Hydrodynamic linear stability of the two-dimensional bluff-body wake through modal analysis and initial-value problem formulation

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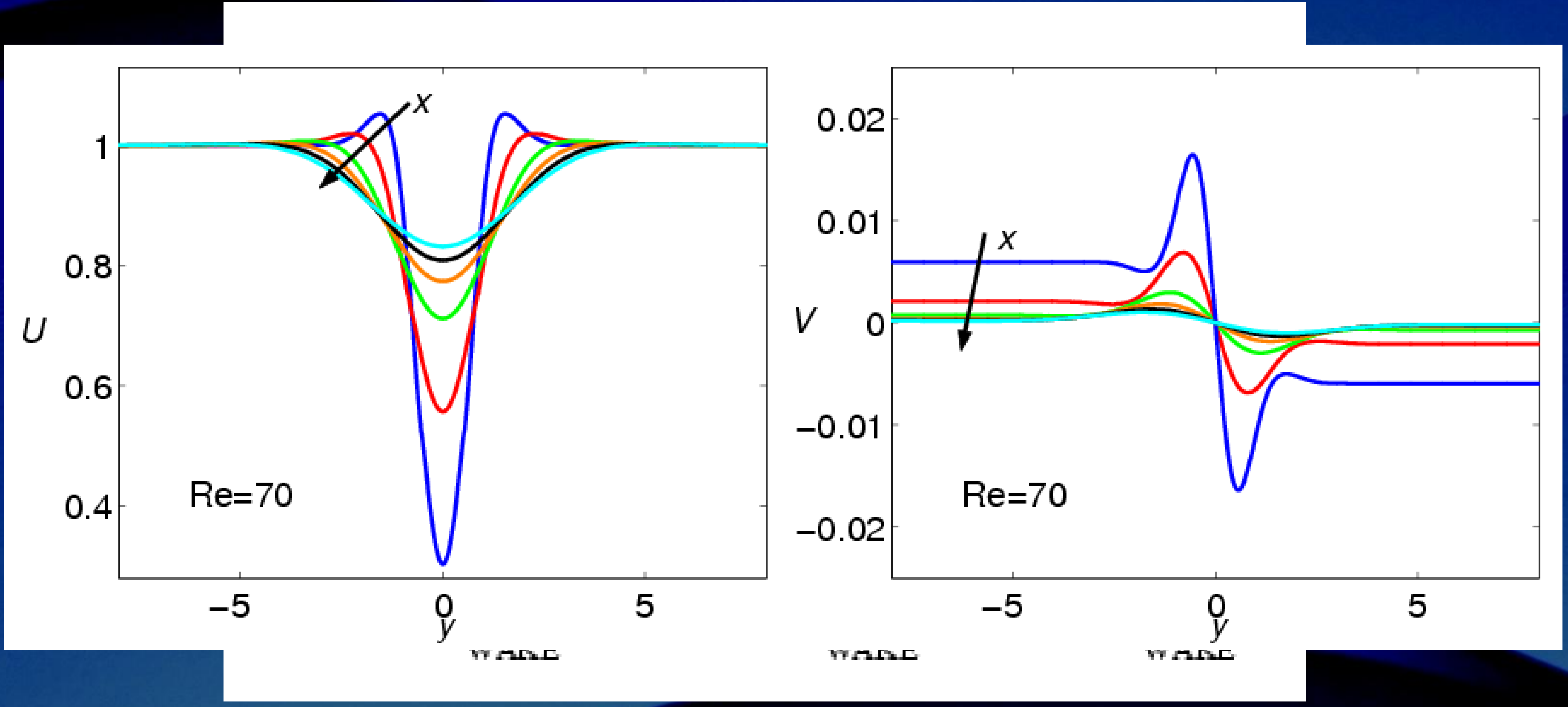
Outline

- Physical problem
- Normal mode analysis
- Entrainment evolution
- Initial-value problem
- Multiscale analysis for the stability of long waves
- Conclusions

Physical Problem

- Flow behind a circular cylinder ➡ steady, incompressible and viscous;
- Approximation of 2D asymptotic Navier-Stokes expansions (Belan & Tordella, 2003), $20 \le Re \le 100$.



Normal mode analysis

The linearized perturbative equation in terms of stream function $\psi(x,y,t)$ is

$$\partial_t \nabla^2 \psi + (\partial_x \nabla^2 \Psi)\psi_y + \Psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \Psi)\psi_x - \Psi_x \partial_y \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi$$

Normal mode hypothesis ⟶ $\psi(x,y,t) = \varphi(x,y,t)\, \mathrm{e}^{i(h_0 x - \sigma_0 t)}$

- $\varphi(x,y,t)$ complex eigenfunction

- $h_0 = k_0 + i\, s_0$ complex wave number
- $\sigma_0 = \omega_0 + i\, r_0$ complex frequency

k_0: wave number
s_0: spatial growth rate
ω_0: frequency
r_0: temporal growth rate

Convective instability: $r_0 < 0$ for all modes, $s_0 < 0$ for at least one mode.

Absolute instability: $r_0 > 0$, $v_g = \partial\sigma_0 / \partial h_0 = 0$ for at least one mode.

Stability analysis through multiscale approach

- Slow variables: $x_1 = \varepsilon x$, $t_1 = \varepsilon t$, $\varepsilon = 1/Re$.
- Hypothesis: $\psi(x,y,t)$ and $\Psi(x,y)$ are expansions in terms of ε:

(ODE dependent on φ_0) + ε (ODE dependent on φ_0, φ_1) + $O(\varepsilon^2)$

Order zero theory Homogeneous Orr-Sommerfeld equation

$$\begin{cases} \mathcal{A}\varphi_0 = \sigma_0 \mathcal{B}\varphi_0 \\ \varphi_0 \to 0,\ |y| \to \infty \\ \partial_y \varphi_0 \to 0,\ |y| \to \infty \end{cases}$$

$$\mathcal{A} = (\partial_y^2 - h_0^2)^2 - ih_0 Re[u_0(\partial_y^2 - h_0^2) - \partial_y^2 u_0]$$

$$\mathcal{B} = -iRe(\partial_y^2 - h_0^2)$$

⟶ eigenfunctions φ_0 and a discrete set of eigenvalues σ_{0n}

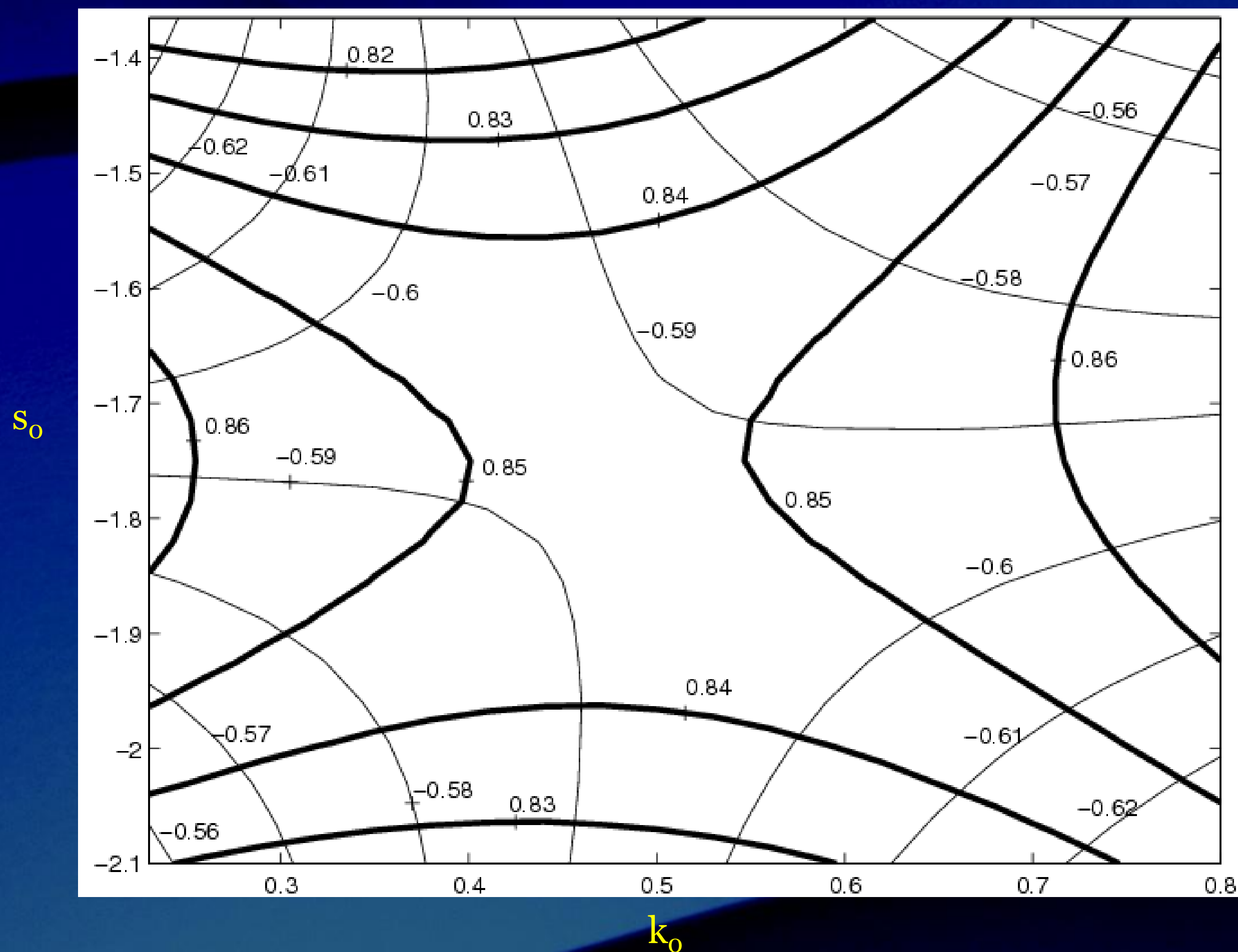
First order theory Non homogeneous Orr-Sommerfeld equation

$$\begin{cases} \mathcal{A}\varphi_1 = \sigma_0 \mathcal{B}\varphi_1 + \mathcal{M}\varphi_0 \\ \varphi_1 \to 0,\ |y| \to \infty \\ \partial_y \varphi_1 \to 0,\ |y| \to \infty \end{cases}$$

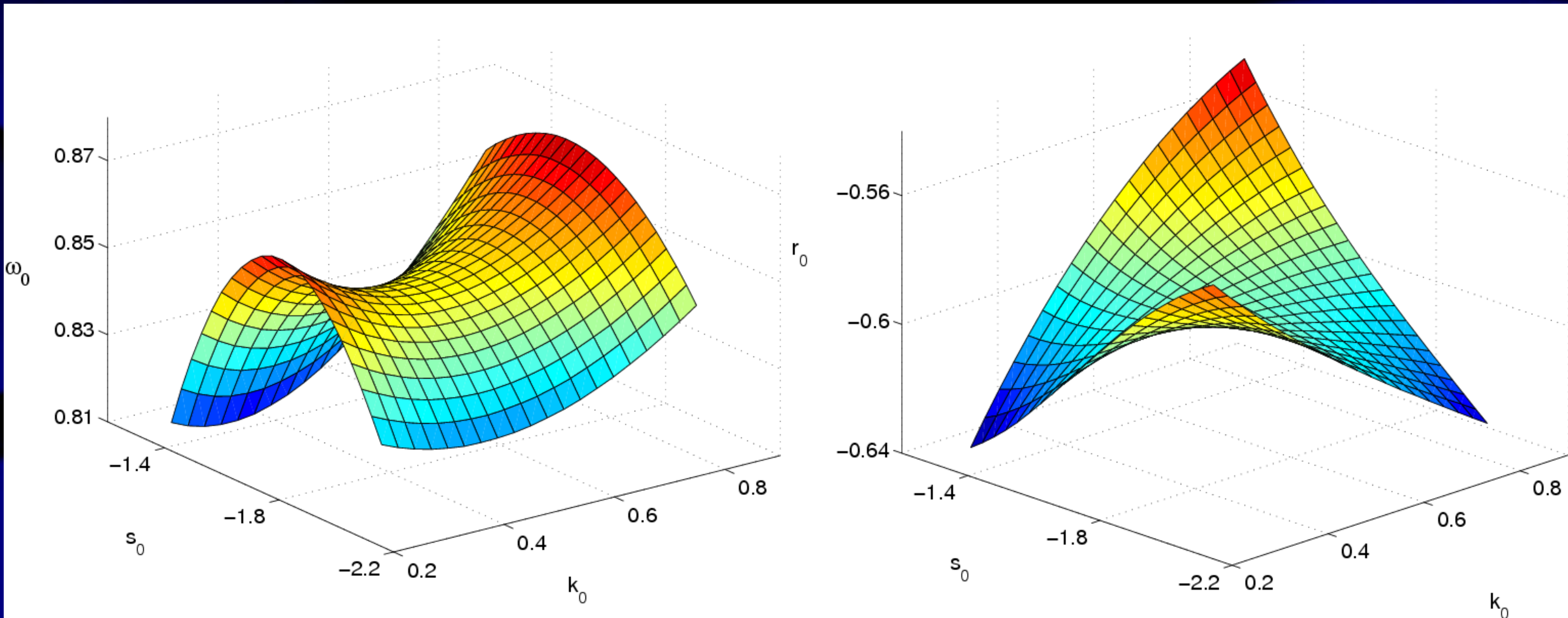
$$\begin{aligned} \mathcal{M} = \{ & \left[Re(2h_0\sigma_0 - 3h_0^2 u_0 - \partial_y^2 u_0) + 4ih_0^3\right] \partial_{x_1} \\ & + (Re u_0 - 4ih_0)\partial^3_{x_1 yy} - Re v_1(\partial_y^3 - h_0^2 \partial_y) + Re \partial_y^2 v_1 \partial_y \\ & + ih_0 Re \left[u_1(\partial_y^2 - h_0^2) - \partial_y^2 u_1\right] + Re(\partial_y^2 - h_0^2)\partial_{t_1} \} \end{aligned}$$

Perturbative hypothesis – Saddle points sequence

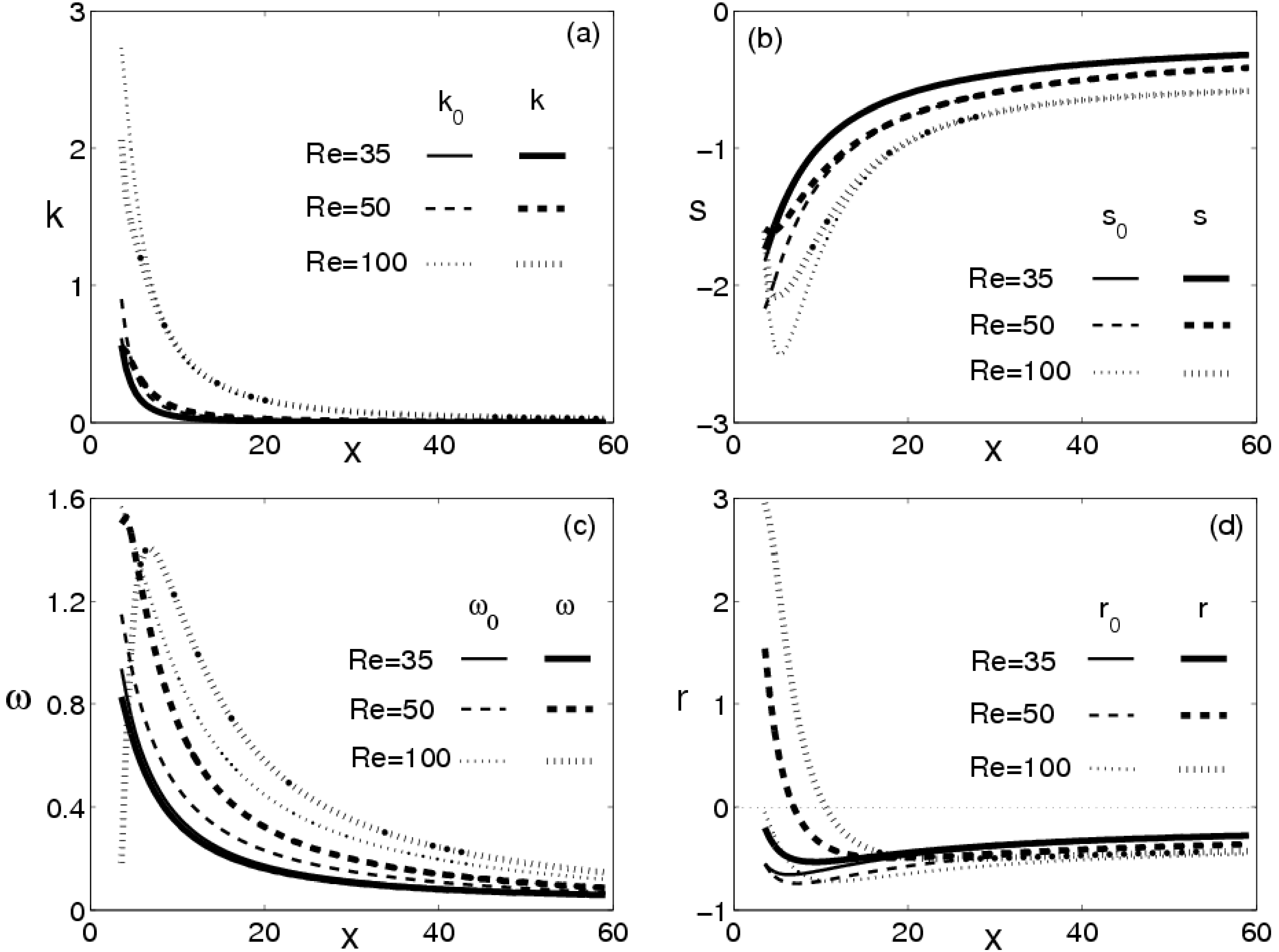
- For fixed values of x and Re the saddle points (h_{0s}, σ_{0s}) of the dispersion relation $\sigma_0=\sigma_0(h_0, x, Re)$ satisfy the condition $\partial\sigma_0/ \partial h_0 = 0$;
- The system is perturbed at every station with the most unstable characteristics at order zero.



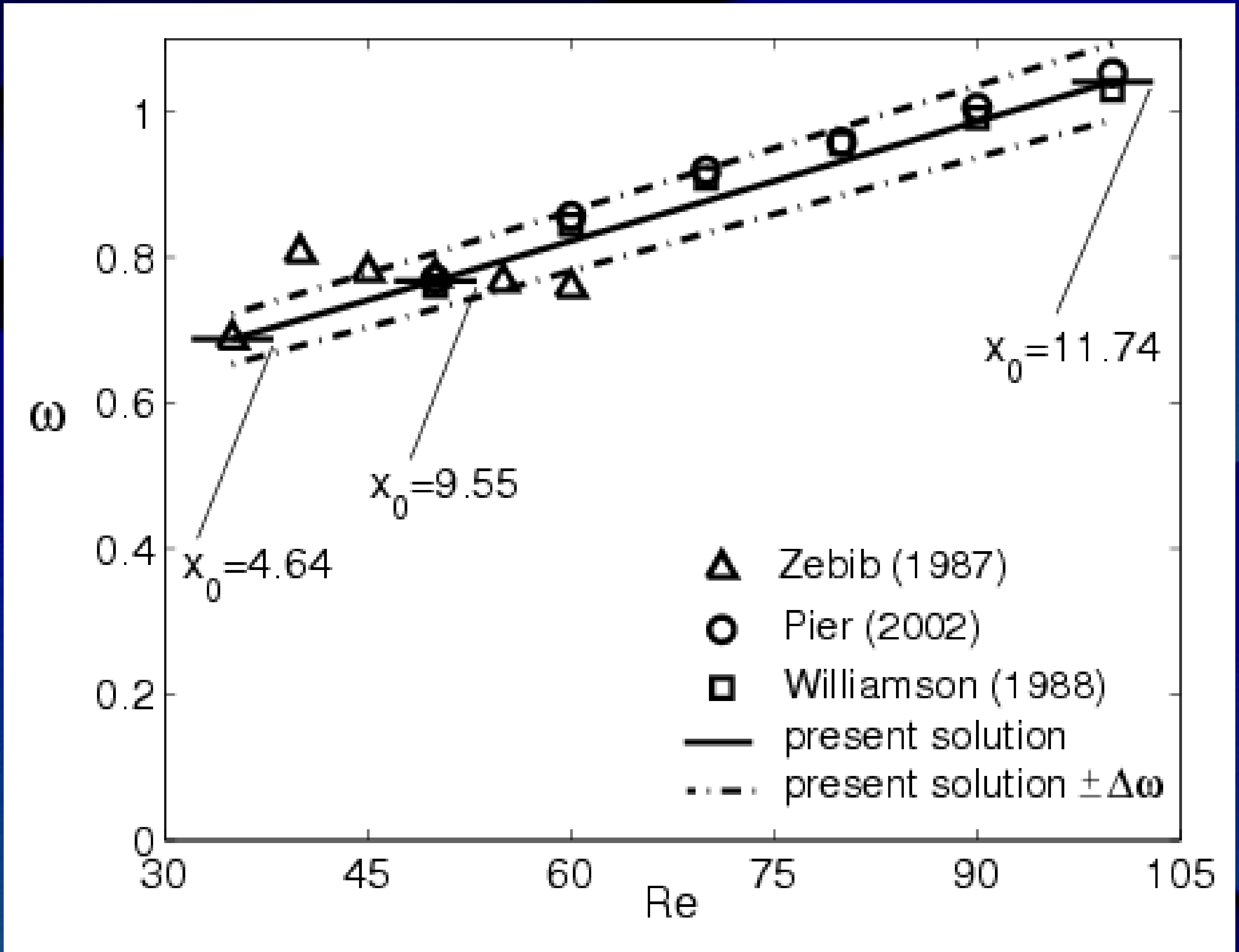
Re=35, x/D=4. Level curves, ω_0=cost (thick curves), r_0=cost (thin curves).



$\omega_0(k_0, s_0)$, $r_0(k_0, s_0)$. Re = 35, x/D = 4.


(a)
k
x
k_0
k
Re=35
Re=50
Re=100
0
1
2
3
0
20
40
60
(b)
s
x
s_0
s
Re=35
Re=50
Re=100
0
−1
−2
−3
0
20
40
60
(c)
ω
x
ω_0
ω
Re=35
Re=50
Re=100
0
0.4
0.8
1.2
1.6
0
20
40
60
(d)
r
x
r_0
r
Re=35
Re=50
Re=100
−1
0
1
2
3
0
20
40
60

Frequency. Comparison between present solution (accuracy $\Delta\omega = 0.05$), Zebib's numerical study (1987), Pier's direct numerical simulations (2002), Williamson's experimental results (1988) .



Tordella, Scarsoglio & Belan, Phys. Fluids 2006.

Eigenfunctions and eigenvalues asymptotic theory

An asymptotic analysis for the Orr-Sommerfeld zero order problem is proposed. For x → ∞ the eigenvalue problem becomes

$$\left\{\partial_y^2 - h_0^2 - ih_0 Re\, u_0\right\} f = -iRe\sigma_0 f$$
$$f \to 0 \text{ as } |y| \to \infty$$

where $f(x,y) = (\partial_y^2 - h_0^2)\varphi_0(x,y)$

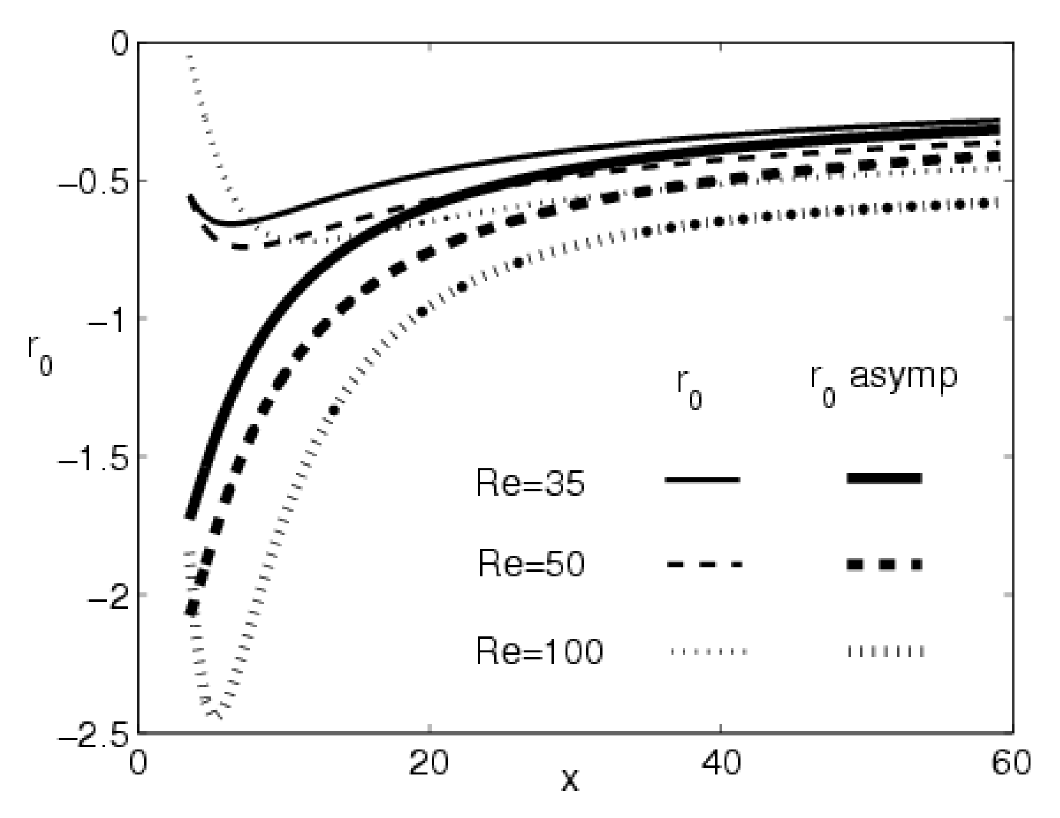
$k_0 \sim 0, \text{ as } x \longrightarrow \infty$

$s_0 < 0, \quad \forall x$

⬇

$\omega_0 \sim 0,\ r_0, s_0 \longrightarrow 0, \text{ as } x \longrightarrow \infty$

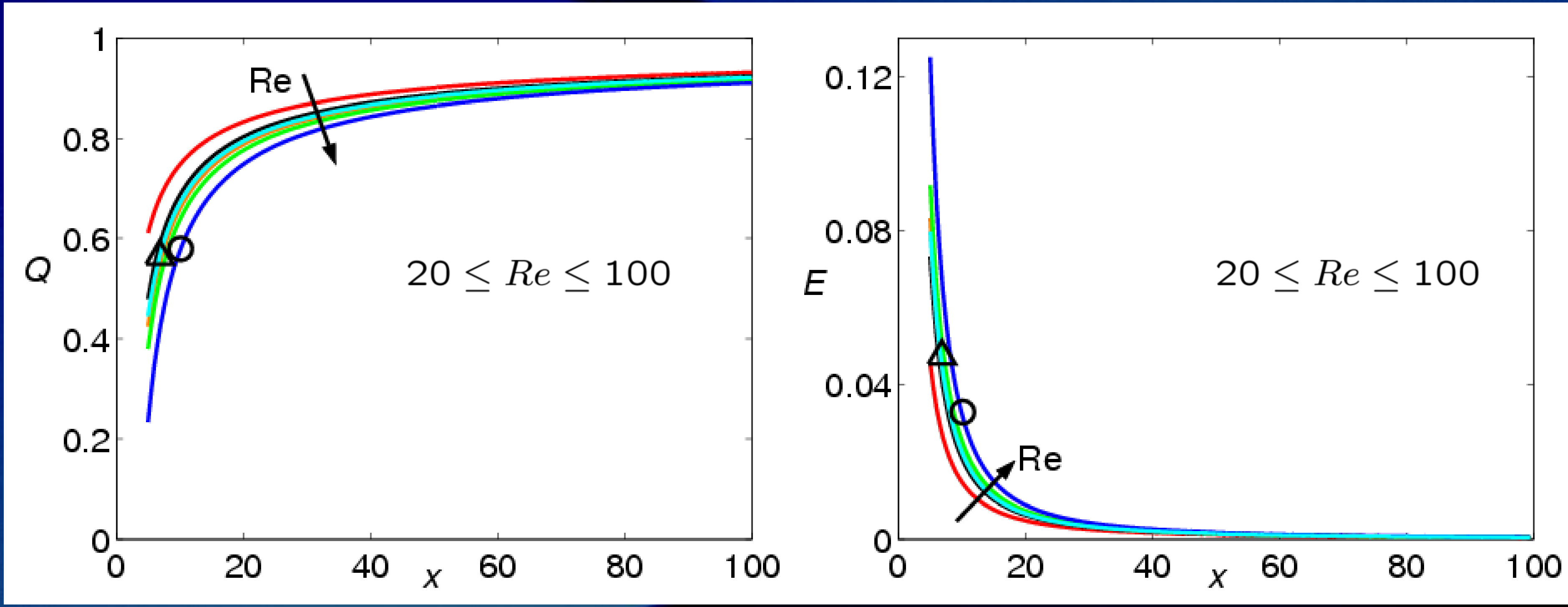
$r_0 \sim s_0 + s_0^2/Re, \text{ as } x \longrightarrow \infty$



Entrainment evolution

$$Q(x) = \frac{1}{2z_w\delta}\int_{-z_w}^{z_w}\int_0^{\delta} U(x,y)\,dy\,dz$$ Volumetric flow rate

$$E(x) = \frac{dQ(x)}{dx}$$ Entrainment



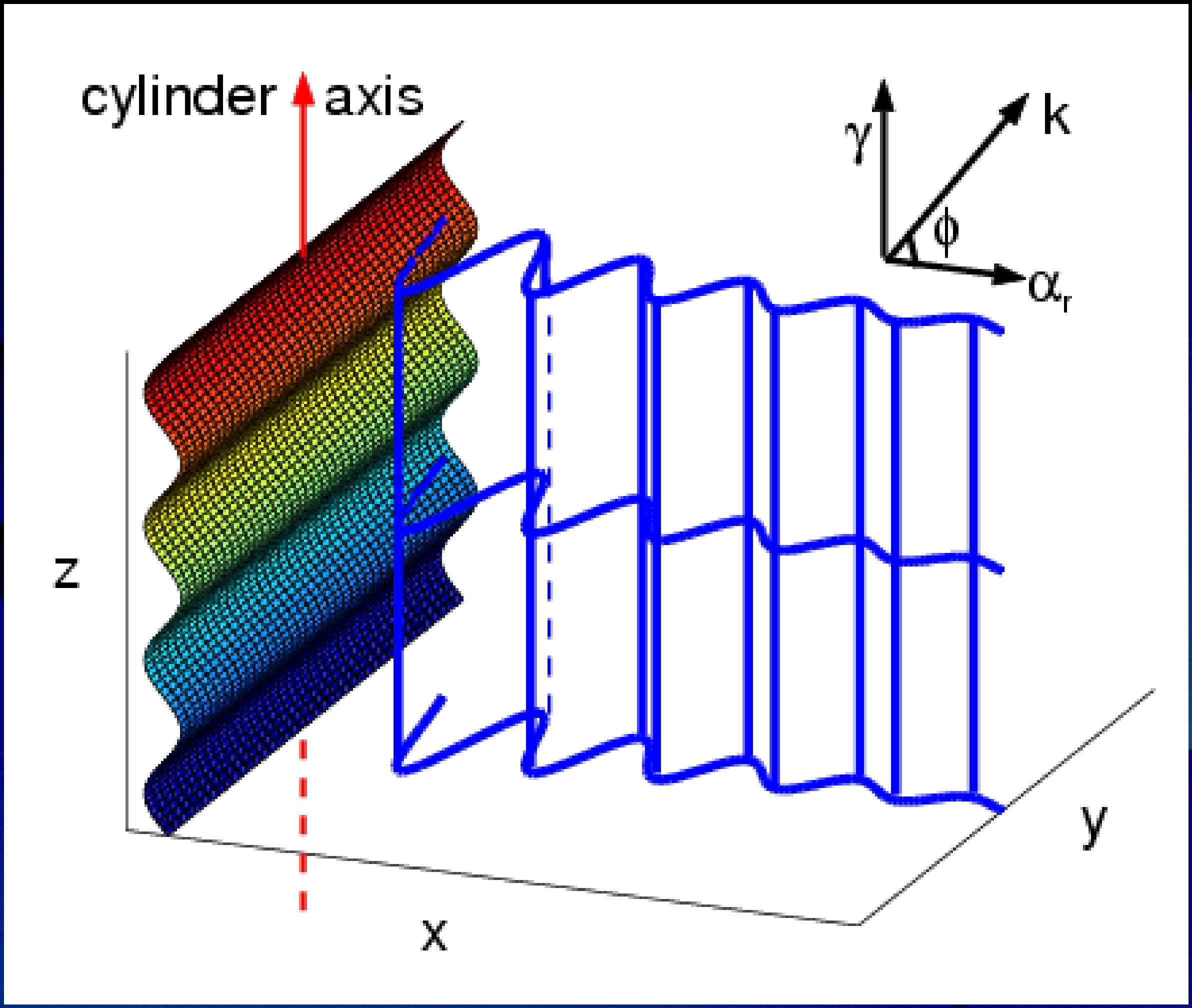
Initial-value problem

- Linear, three-dimensional perturbative equations in terms of vorticity and velocity (Criminale & Drazin, 1990);
- Base flow parametric in *x* and *Re* ➡ U(y; x_0, Re)
- Laplace-Fourier transform in x and z directions for perturbation quantities:

$$
\begin{cases}
\dfrac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2ikcos(\phi)\alpha_i)\hat{v} = \hat{\Gamma} \\
\dfrac{\partial \hat{\Gamma}}{\partial t} = -(ikcos(\phi) - \alpha_i)U\hat{\Gamma} + (ikcos(\phi) - \alpha_i)\dfrac{d^2U}{dy^2}\hat{v} \\
\quad + \dfrac{1}{Re}[\dfrac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2ikcos(\phi)\alpha_i)\hat{\Gamma}] \\
\dfrac{\partial \hat{\omega}_y}{\partial t} = -(ikcos(\phi) - \alpha_i)U\hat{\omega}_y - iksin(\phi)\dfrac{dU}{dy}\hat{v} \\
\quad + \dfrac{1}{Re}[\dfrac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2ikcos(\phi)\alpha_i)\hat{\omega}_y]
\end{cases}
$$

$$\tilde{\omega}_y = \frac{\partial \tilde{u}}{\partial z} - \frac{\partial \tilde{w}}{\partial x}$$

$$\tilde{\Gamma} = \frac{\partial \tilde{\omega}_z}{\partial x} - \frac{\partial \tilde{\omega}_x}{\partial z}$$



$a_r = k\ cos(\Phi)$ wavenumber in x-direction

$\gamma = k\ sin(\Phi)$ wavenumber in z-direction

$\Phi = tan^{-1}(\gamma / a_r)$ angle of obliquity

$k = (a_r^2 + \gamma^2)^{1/2}$ polar wavenumber

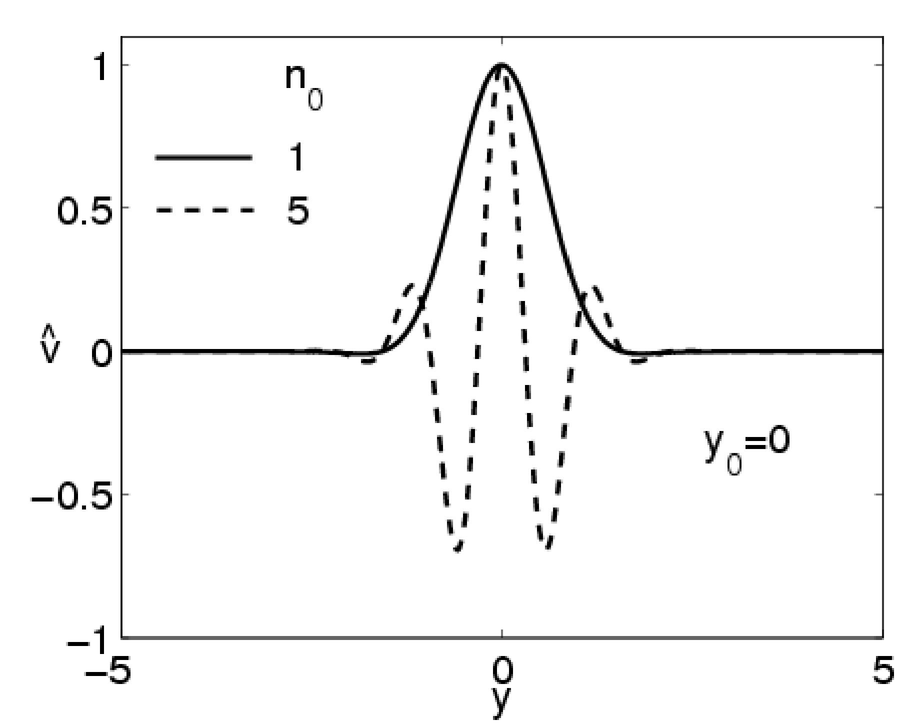
$a_i \geq 0$ spatial damping rate

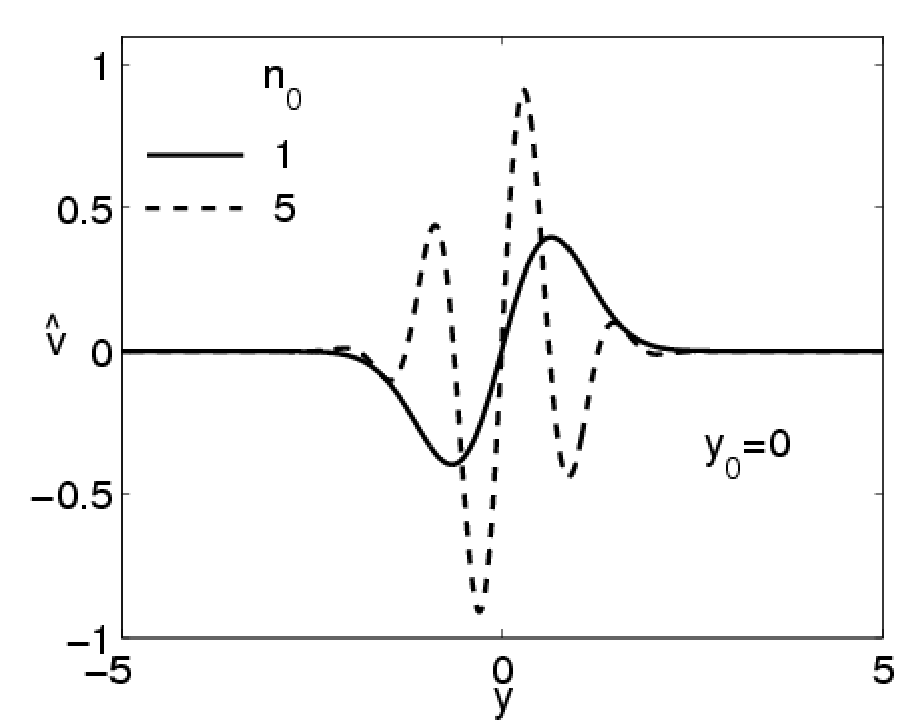
- Periodic initial conditions for $\hat{\Gamma} = \frac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2ikcos(\phi)\alpha_i)\hat{v}$

$$\begin{cases} \hat{v}(y, t=0) = e^{-(y-y_0)^2} cos(n_0(y-y_0)) & \text{symmetric} \\ \hat{v}(y, t=0) = e^{-(y-y_0)^2} sin(n_0(y-y_0)) & \text{asymmetric} \end{cases}$$

and $\hat{\omega}_y(y, t=0) = 0$

- Velocity field vanishing in the free stream.





Early transient and asymptotic behaviour of perturbations

- The growth function G is the normalized kinetic energy density

$$G(t;\alpha,\gamma) = \frac{e(t;\alpha,\gamma)}{e(t=0;\alpha,\gamma)}$$

and measures the growth of the perturbation energy at time t.

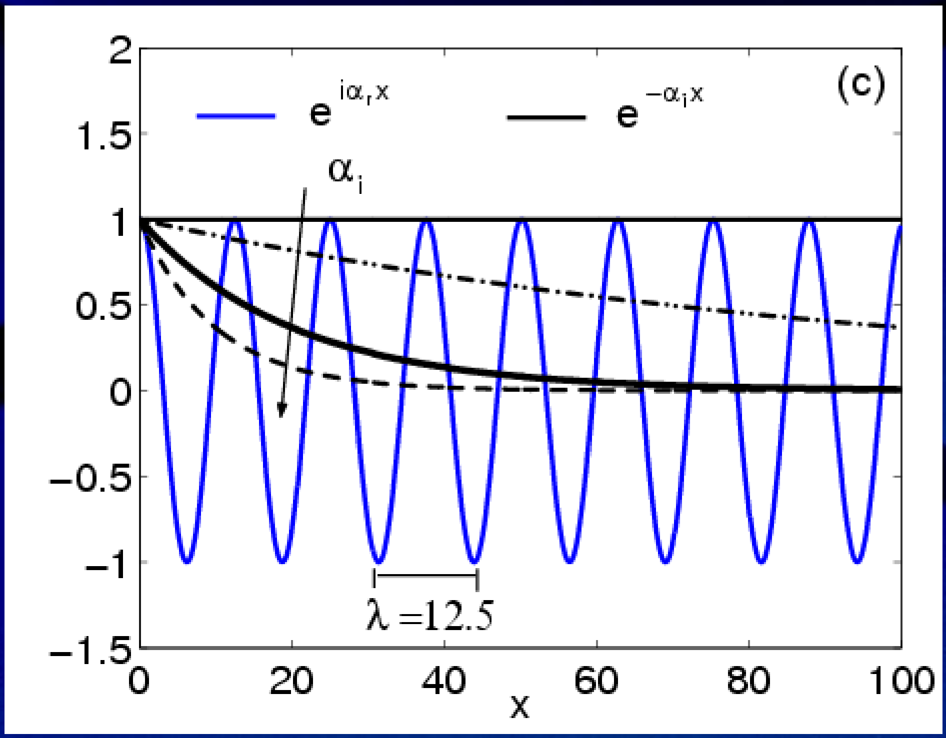
- The temporal growth rate r (Lasseigne et al., 1999) and the angular frequency ω (Whitham, 1974)

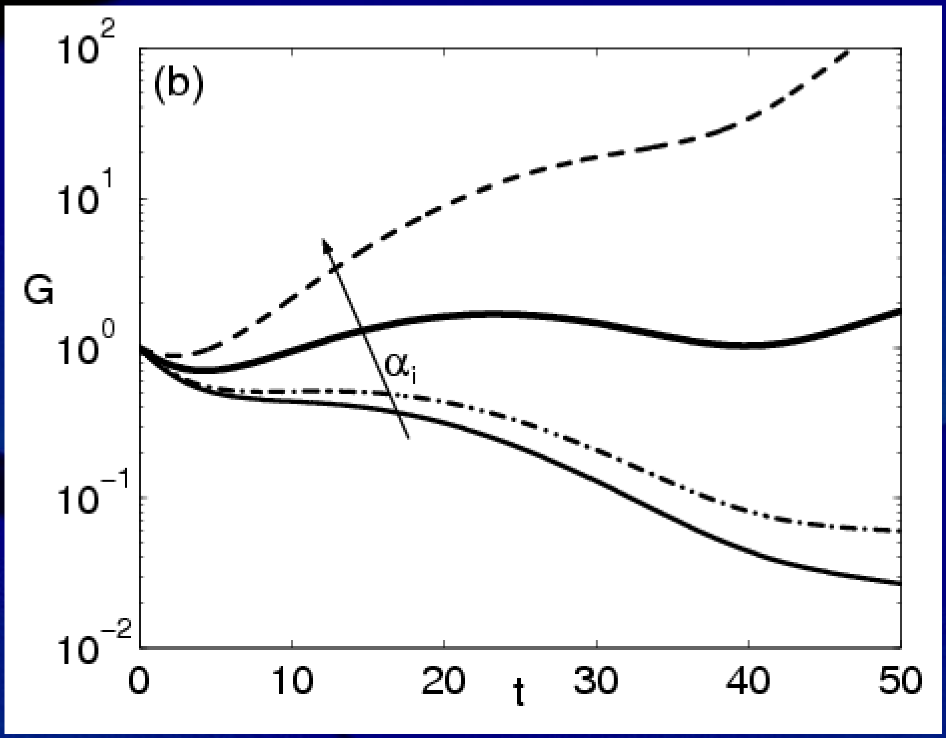
$$r(t;\alpha,\gamma) = \frac{log|e(t;\alpha,\gamma)|}{2t}, \quad t > 0$$

$$\omega(t;\alpha,\gamma) = \frac{|d\varphi(t;\alpha,\gamma)|}{dt}$$

φ perturbation phase

Exploratory analysis of the transient dynamics

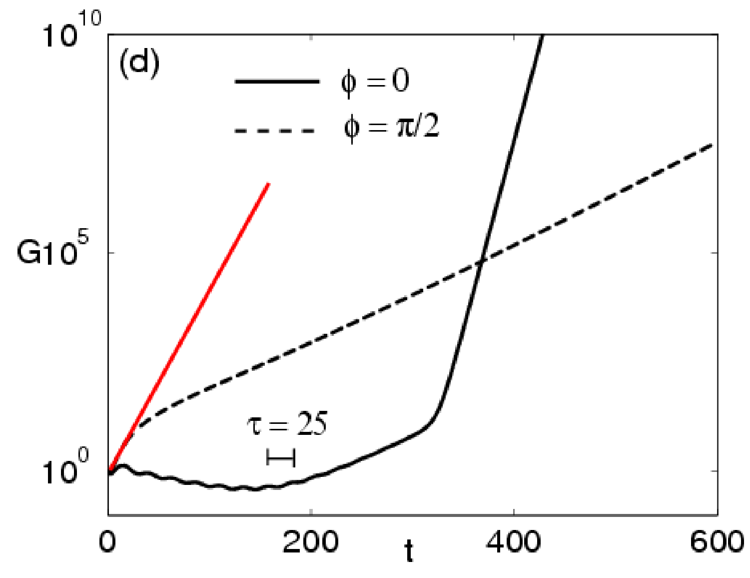




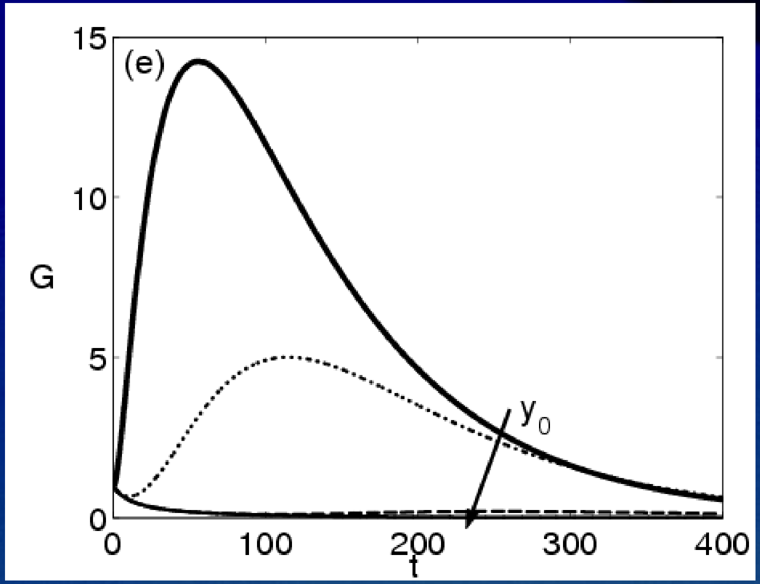
(a): Wave spatial evolution in the x direction, for asymmetric, $\Phi=3/8\pi$, k=0.5,1,1.5,2,2.5.

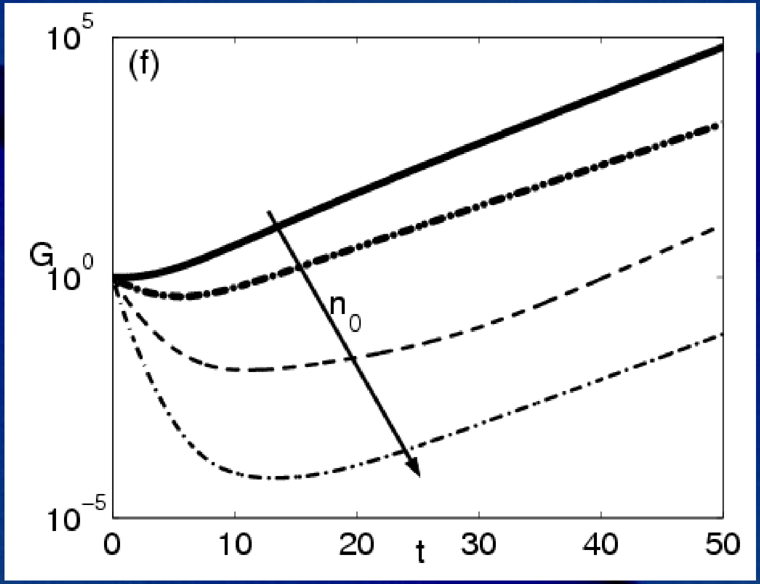
(c): R=100, $y_0=0$, $x_0=6.50$, $\alpha_i=0.01$, $n_0=1$, $k=\alpha_r=0.5$, $\alpha_i=0,0.01,0.05,0.1$.

(b): R=50, $y_0=0$, $x_0=7$, k=0.5, $\Phi=0$, asymmetric, $n_0=1$, $\alpha_i = 0,0.01,0.05,0.1$.



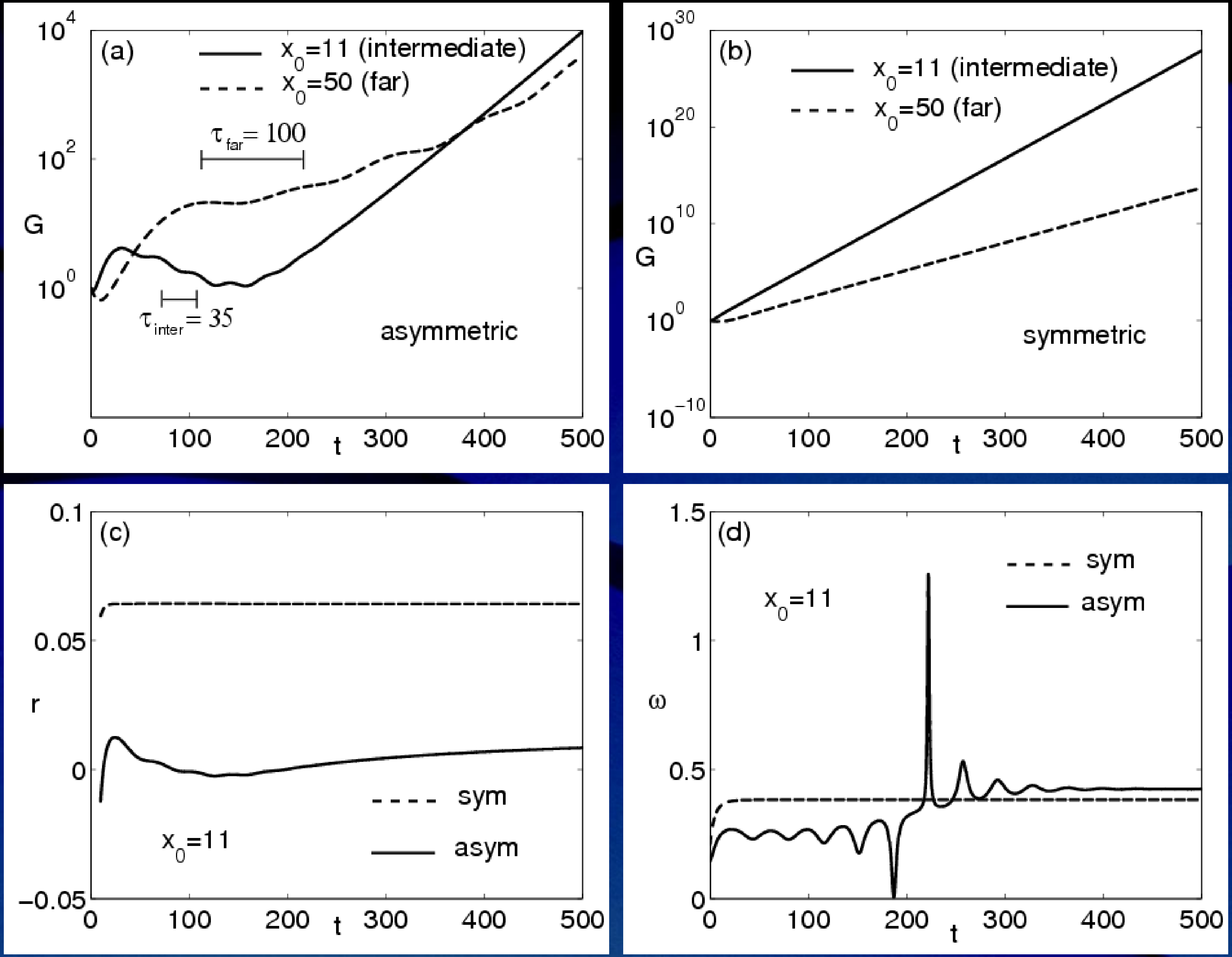
(d): R=100, y_0=0, x_0=11.50, k=0.7, asymmetric, α_i=0.02, n_0=1, Φ=0, $\pi/2$.





(e): R=100 x_0=12, k=1.2, α_i=0.01, symmetric, n_0=1, Φ= $\pi/2$, y_0=0,2,4,6.

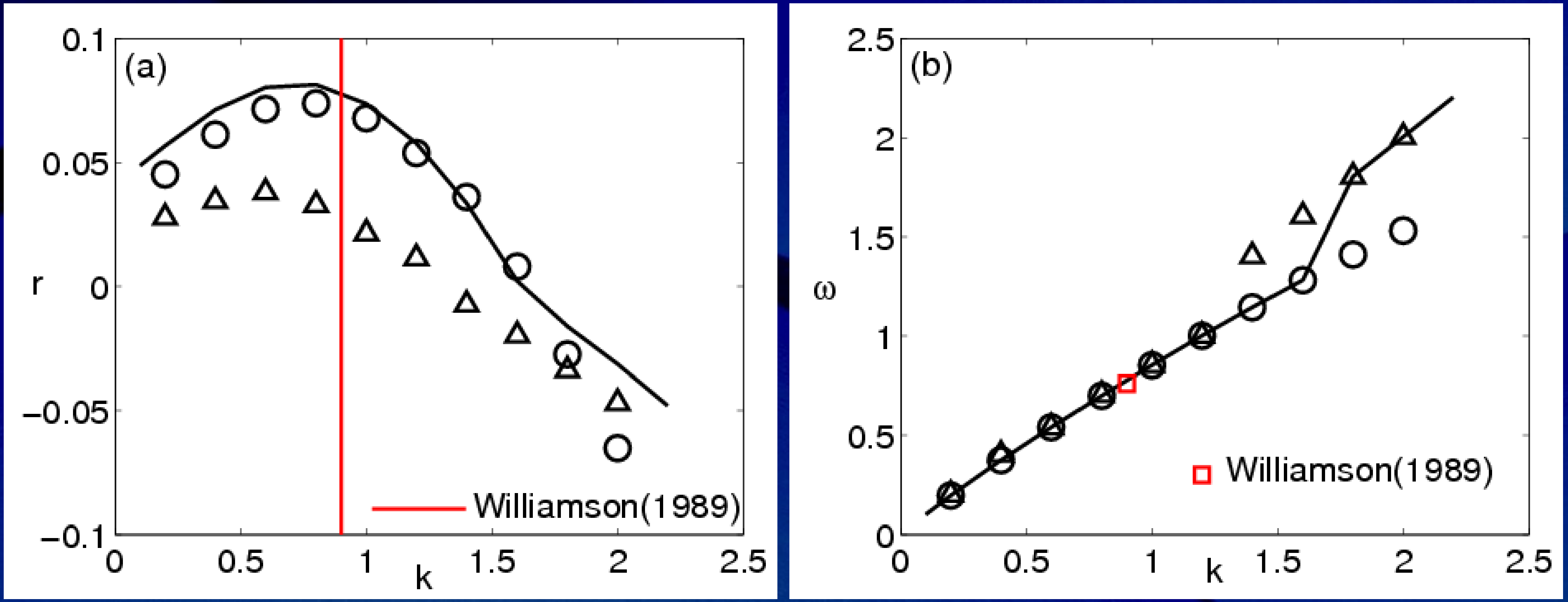
(f): R=50 x_0=14, k=0.9, α_i=0.15, asymmetric, y_0=0, Φ= $\pi/2$, n_0=1,3,5,7.



(a)-(b)-(c)-(d): R=100, y_0=0, k=0.6, α_i=0.02, n_0=1, $\Phi=\pi/4$, x_0=11 and 50, symmetric and asymmetric.

Asymptotic fate and comparison with modal analysis

- Asymptotic state: the temporal growth rate r asymptotes to a constant value ($dr/dt < \varepsilon \sim 10^{-4}$).



(a)-(b): Re=50, α_i=0.05, Φ=0, x_0=11, n_0=1, y_0=0. Initial-value problem (triangles: symmetric, circles: asymmetric), normal mode analysis (black curves), experimental data (Williamson 1989, red symbols).

Multiscale analysis for the stability of long waves

- Different scales in the stability analysis:
  - Slow scales (base flow evolution);
  - Fast scales (disturbance dynamics);

- In some flow configurations, long waves can be destabilizing (for example Blasius boundary layer and 3D cross flow boundary layer);

- In such instances the perturbation wavenumber of the unstable wave is much less than O(1).

Small parameter is the polar wavenumber of the perturbation:

$$k \ll 1$$

Full linear system

$$\begin{cases} \dfrac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2ikcos(\phi)\alpha_i)\hat{v} = \hat{\Gamma} \\ \dfrac{\partial \hat{\Gamma}}{\partial t} = G\hat{\Gamma} + H\hat{v} + K\hat{\omega}_y \\ \dfrac{\partial \hat{\omega}_y}{\partial t} = L\hat{\omega}_y + M\hat{v} \end{cases}$$

$$G = G(y; k, \phi, \alpha_i, Re)$$

base flow

(U(x,y:Re), V(x,y;Re))

Multiple scales hypothesis

- Regular perturbation scheme, k<<1:

$$\begin{aligned} \hat{v} &= \hat{v}_0 + k\hat{v}_1 + k^2\hat{v}_2 + \ldots \\ \hat{\Gamma} &= \hat{\Gamma}_0 + k\hat{\Gamma}_1 + k^2\hat{\Gamma}_2 + \ldots \\ \hat{\omega}_y &= \hat{\omega}_{y0} + k\hat{\omega}_{y1} + k^2\hat{\omega}_{y2} + \ldots \end{aligned}$$

- Temporal scales: $t, \tau = kt, T = k^2 t;$
- Spatial scales: $y, Y = ky;$

Order O(1)

$$\left\{\begin{array}{rcl} \frac{\partial^2 \hat{v}_0}{\partial y^2} + \alpha_i^2 \hat{v}_0 & = & \hat{\Gamma}_0 \\ \frac{\partial \hat{\Gamma}_0}{\partial t} - G_h \hat{\Gamma}_0 - H_h \hat{v}_0 & = & 0 \\ \frac{\partial \hat{\omega}_{y0}}{\partial t} - L_h \hat{\omega}_{y0} & = & 0 \end{array}\right.$$

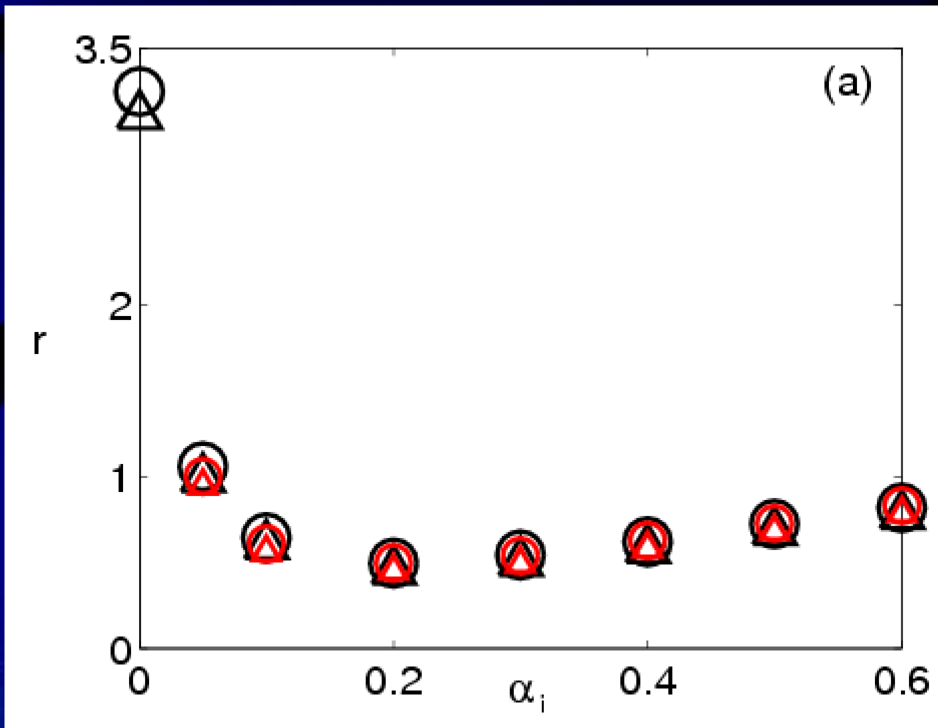
$$G_h = G_h(y; \phi, \alpha_i, Re)$$

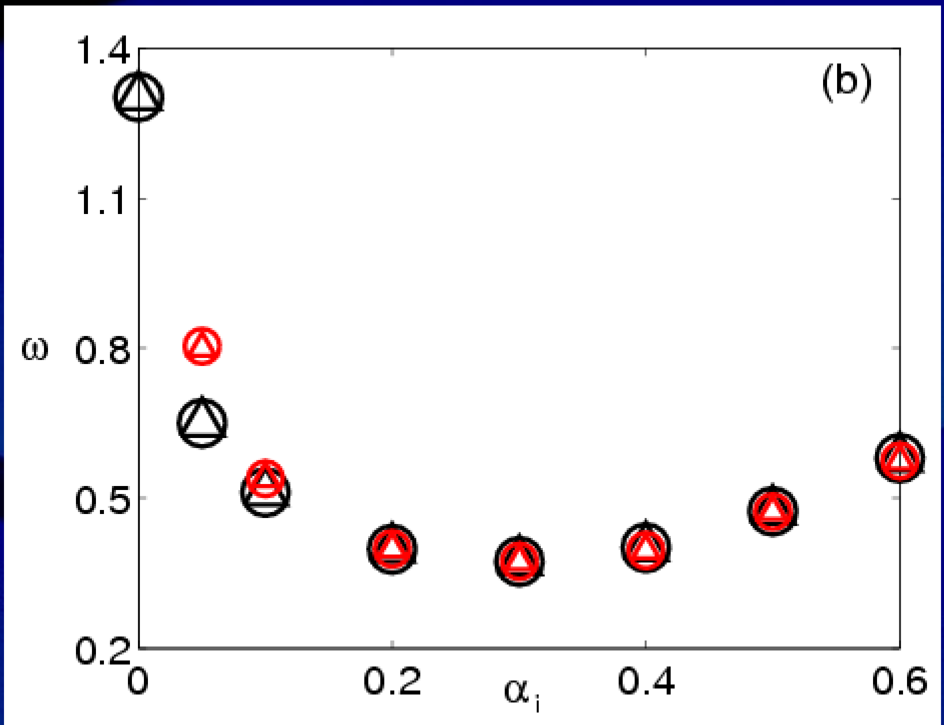
Order O(k)

$$\left\{\begin{array}{rcl} \frac{\partial^2 \hat{v}_1}{\partial y^2} + \alpha_i^2 \hat{v}_1 & = & -2\frac{\partial^2 \hat{v}_0}{\partial y \partial Y} + 2icos(\phi)\alpha_i \hat{v}_0 + \hat{\Gamma}_1 \\ \frac{\partial \hat{\Gamma}_1}{\partial t} - G_h \hat{\Gamma}_1 - H_h \hat{v}_1 & = & -\frac{\partial \hat{\Gamma}_0}{\partial \tau} + G_{h-1}\hat{\Gamma}_0 + H_{h-1}\hat{v}_0 + K_{h-1}\hat{\omega}_{y0} \\ \frac{\partial \hat{\omega}_{y1}}{\partial t} - L_h \hat{\omega}_{y1} & = & -\frac{\partial \hat{\omega}_{y0}}{\partial \tau} + L_{h-1}\hat{\omega}_{y0} + M_{h-1}\hat{v}_0 \end{array}\right.$$

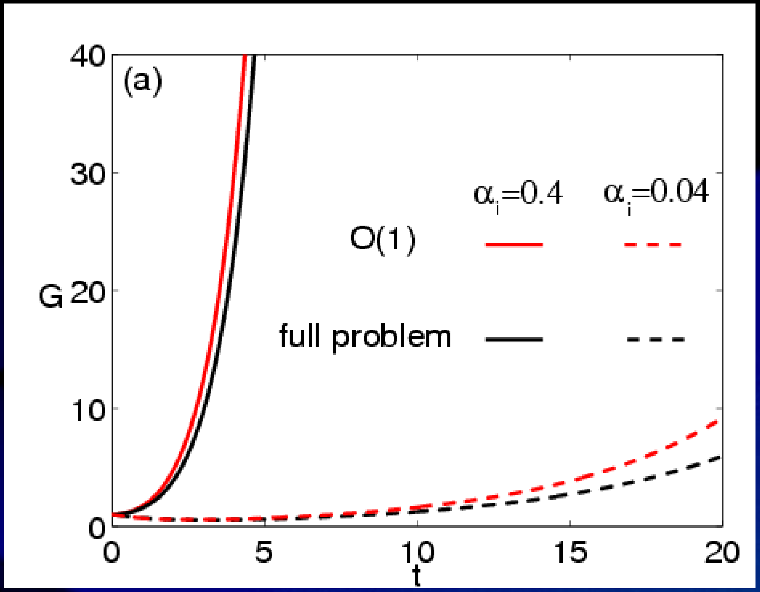
$$G_{h-1} = G_{h-1}(y, Y; \phi, \alpha_i, Re)$$

Comparison with the full linear problem

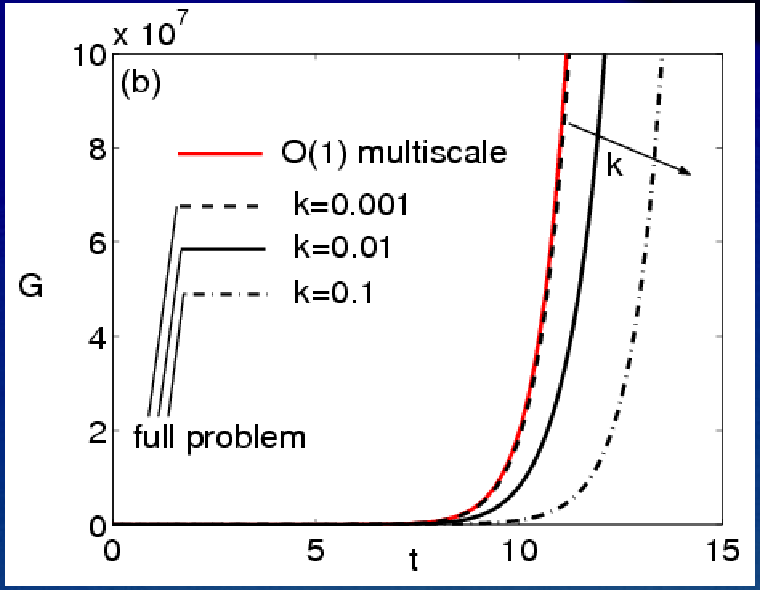


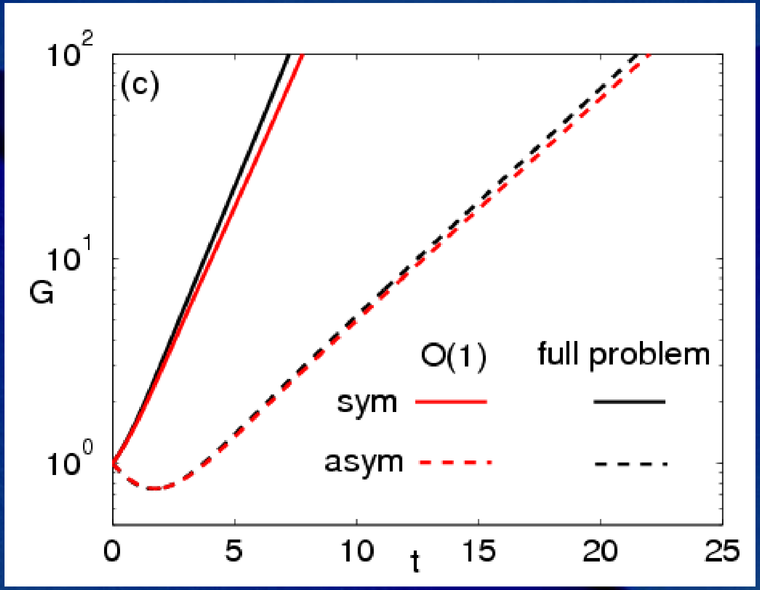


(a)-(b): Re=100, k=0.01, $\Phi=\pi/4$, $x_0=10$, $n_0=1$, $y_0=0$. Full linear problem (black circles: symmetric, black triangles: asymmetric), multiscale O(1) (red circles: symmetric, red triangles: asymmetric).



(a): R=50, y_0=0, k=0.03, n_0=1, x_0=12, Φ=π/4, asymmetric, α_i=0.04, 0.4.





(b): R=100, y_0=0, n_0=1, x_0=27, Φ=0, symmetric, α_i=0.2, k=0.1, 0.01, 0.001.

(c): R=100, y_0=0, k=0.02, x_0=13.50, n_0=1, Φ=π/2, α_i=0.08, sym and asym.

Conclusions

- Synthetic perturbation hypothesis (saddle point sequence);
- Absolute instability pockets (Re=50,100) found in the intermediate wake;
- Good agreement, in terms of frequency, with numerical and experimental data;
- *No information on the early time history of the perturbation;*
- Different transient growths of energy;
- Asymptotic good agreement with modal analysis and with experimental data (in terms of frequency and wavelength);
- Multiscaling O(1) for long waves well approximates full linear problem.