Hydrodynamics linear stability theory. A comparison between Orr-Sommerfeld modal and initial value problem analyses

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Abstract. *We consider a classical problem of hydrodynamics instability, the two-dimensional bluff-body wake, to compare the outcomes that the contemporary Orr-Sommerfeld modal analysis and the initial value problem analysis can offer. The steady wake is described in a non conventional way by taking into account its full dynamics. In fact, these two approaches are used by taking into account the nonlinear nonparallel and diffusive nature of the basic flow. It is shown that the insertion of the transversal dynamics in the perturbative equations of the modal theory allows to get stability characteristics and critical values of the flow control parameter that closely meet the laboratory results. Furthermore, the great variety of the early and intermediate transient behaviour of small three-dimensional perturbations is presented. As far as the perturbation asymptotic fate is concerned, the equivalence of two stability analyses can be demonstrated.*

Keywords: convective and absolute instability, basic sheared flow, nonparallel, initial value problem, early transient.

Riassunto. *I risultati offerti dall'odierna teoria modale di Orr-Sommerfeld e dall'analisi secondo il problema ai valori iniziali sono confrontati per mezzo di un classico problema di instabilità idrodinamica, la scia bidimensionale generata da un corpo tozzo. La configurazione di equilibrio stazionario è qui descritta in modo non classico, tenendo conto cioè del modello fisico completo, tradizionalmente molto semplificato nell'ambito degli studi sull'instabilità lineare. Infatti, in questo lavoro, la natura diffusiva e gli aspetti non lineari e di non parallelismo del flusso base sono esplicitamente inseriti in entrambe le formulazioni. L'introduzione della dinamica trasversale all'interno dell'analisi modale permette di ottenere risultati consistenti con le premesse e*

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finalmente inoltre un ottimo accordo con i dati sperimentali. Si presenta inoltre un'ampia varietà di transitori nel breve ed intermedio termine che la formulazione ai valori iniziali associata a perturbazioni tridimensionali riesce a determinare. E' possibile dimostrare che le due analisi conducono asintoticamente a risultati equivalenti.

Parole chiave: instabilità convettiva e assoluta, flusso di taglio, non-parallelismo, problema ai valori iniziali, transitorio iniziale.

1. Introduction

Hydrodynamic stability is most important in the fields of aerodynamics, hydromechanics, combustion, oceanography, atmospheric dynamics, astrophysics, and biology. A point of universality in these natural and technical contexts is that the laminar flow configuration is exceptionally met. The breakdown of laminar flow has been a central issue for over a hundred years and it is still an open problem because a commonly recognized mean of prediction is yet to be defined. The main point at issue is that the stability or instability mechanisms determine, to a greater extent, the performance of a system.

Early ideas and investigations about the linear hydrodynamics stability trace back to scientist such as Leonardo da Vinci, Gotthilf Hagen, Osborne Reynolds, Lord Rayleigh and Lord Kelvin. Exactly one hundred years ago, in 1907, William M. F. Orr [10]-[11] published a comprehensive review (129 pages) on the subject in the Proceedings of the Royal Irish Academy, that included for the very first time a very detailed analysis of the stability or instability of the parallel motion of a viscous liquid. One year later, in 1908, Arnold Sommerfeld [15] published in the Proceedings of the fourth International Conference of the Mathematicians held in Rome, a short synthetic paper (6 pages) on the same subject. Since then, the main theoretical tool for the analysis of small perturbations in sheared flow, the famous Orr-Sommerfeld equation of stability theory, was set out. This is a 4th order temporal-spatial partial differential equation that uncouples the velocity variable directed across the sheared steady basic flow from the other velocity variables. By adopting a modal expansion in normal modes, the Orr-Sommerfeld equation takes the familiar form still in use today, that is that of a 4th order ordinary differential homogeneous, but non autonomous, equation for the Fourier-Laplace transform of the transversal velocity variable.

In the present paper, analytical Navier-Stokes expansion solutions are used to approximate the wake profile. The base flow model is physically accurate as it directly includes the transversal nonlinear and diffusive dynamics of the flow. Besides the lateral variations of the transversal velocity in the free stream, the Navier-Stokes model allows the longitudinal and lateral variations of the pressure field to be estimated [1]-[16]. The base flow transversal dynamics is explicitly inserted into the perturbative equations, so that they are now written in a non-homogeneous form. Through the multiple scales approach based on the inverse of the Reynolds number ($\varepsilon = 1/R$), the dispersion relation can be expressed in a synthetic form as it only depends on the Reynolds number R and the longitudinal coordinate x. This is an innovative feature of the present study and can be extended to the stability analysis of other spatially developing flows. The disturbance can have uniform length throughout the spatial domain or can be locally tuned to the most unstable wavenumber (according to dispersion relation distribution along x). Both the perturbative hypotheses lead to an excellent agreement - in terms of perturbation angular frequency - with experimental [20] and numerical [21]-[12] data. Moreover, as the disturbance frequency is a local rather than a global instability characteristic, the wake region where an unstable configuration can hold is identified. For the first time, for $R = 50$ and 100, absolute instability pockets are determined in the early part of the intermediate wake, where the WKBJ method is completely consistent.

For the initial-value problem analysis, the base flow has been obtained approximating it only with the longitudinal component of the Navier-Stokes expansion solution for velocity field that considers the lateral entrainment process. The near-parallel hypothesis for the base flow, at every longitudinal station, is made. The longitudinal coordinate x_0 plays the role, together with the Reynolds number, of parameter of the system. The formulation is carried on in terms of three-dimensional perturbation vorticity and the temporal behaviour, including both the early time transient as well as the long time asymptotics, can be observed [3], [9], [8]. A moving coordinate transformation [5] is adopted and a two-dimensional Laplace-Fourier decomposition is then performed in streamwise and spanwise directions for every perturbation quantity. The introduction of a complex longitudinal wavenumber is an innovative feature as to explicitly include, also in the structure of the perturbation, a degree of freedom associated to the spatial evolution of the system. The resulting partial differential equations in y and t are then numerically solved by the method of lines. Different physical inputs linked to the symmetry, the obliquity and the spatial growth rate of the perturbation are considered to study the early transient. In most of the cases, the initial temporal evolution is not *a priori* predictable, as configurations with initial damping followed by a fast growth or with initial transient growths that smoothly decrease in time are shown. It can be demonstrated that, after the transient is extinguished, the analyses asymptote to the same values.

In §2, the physical problem together with the relevant aspects of the Navier-Stokes expansions approximating the wake profile are introduced. The Orr-Sommerfeld analysis is presented in §3, and details on the multiple scales approach formulation are given in §3.1. The perturbation hypotheses and subsequent results are described in §3.2. The initial-value problem analysis is presented in §4. The basis of the formulation and significant results of the perturbation temporal evolution are given in §4.1 and §4.2, respectively. Concluding remarks are offered in §5.

2. Physical problem

The stability analysis of the 2D bluff body wake can be performed on a steady base flow obtained as an analytical approximated Navier-Stokes solution valid in the intermediate and far field, which includes both the inner high vorticity region and the outer low vorticity region. The solution model fully accounts for the non linearities of the convective transport as well as for the diffusive longitudinal and transversal transport. This is an important point since the bifurcation between the steady and the unsteady flow configuration happens at low values of the Reynolds number (e.g., $R_{cr} = 47$ in case of the flow past a circular indefinite cylinder) [1, 16].

The analytical solution is a matched asymptotic expansion where lower order terms account for the effects of streamwise diffusion, non linear convection and entrainment, and higher order terms account for pressure gradients and vorticity diffusion. Here the body is placed in a reference system with standard adimensional longitudinal and normal coordinates x, y. The adimensionalization is referred to the scale D of the body, the density ρ and the velocity U of the free stream. The Reynolds number is defined as $R = \rho UD/\mu$.

The base flow is obtained by matching an inner NS expansion in powers of the inverse of the longitudinal coordinate ($x^{-n/2}$, $n = 0, 1, 2...$) and an outer NS expansion in powers of the inverse of the distance $r = \sqrt{x^2 + y^2}$ from the body ($r^{-n/2}$, $n = 0, 1, 2...$). The physical quantities involved in matching criteria are the vorticity, the longitudinal pressure gradients and the entrainment velocities. The lateral decay results to be algebraic at high orders in the inner expansion solution.

The inner variables x, η are defined by the quasi-similar transformation

$$x = x,\ \eta = x^{-1/2} y, \tag{1}$$

the relevant velocity components and pressure can be written as

$$u_i = 1 + \sum_n x^{-n/2} \phi_n(\eta)\ ,\ v_i = \sum_n x^{-n/2} \chi_n(\eta)\ ,\ p_i = \pi_0 + \sum_n x^{-n/2} \pi_n(\eta). \tag{2}$$

The outer variables r, s are defined by the transformation

$$r = \sqrt{x^2 + y^2},\ s = y/x\,, \tag{3}$$

the relevant velocity components and pressure can be written as

$$u_o = 1 + \sum_n r^{-n/2} f_n(s)\ ,\ v_o = \sum_n r^{-n/2} g_n(s)\ ,\ p_o = p_0 + \sum_n r^{-n/2} p_n(s). \tag{4}$$

The matching conditions are given for each x along the wake, they relate logitudinal pressure gradients, vorticities and transverse velocities:

$$\lim_{y\to 0} \partial_x p_o = \lim_{y\to\infty} \partial_x p_i, \tag{5}$$

$$\lim_{y\to 0} \omega_o = \lim_{y\to\infty} \omega_i, \tag{6}$$

$$\lim_{y\to 0} v_o = \lim_{y\to\infty} v_i. \tag{7}$$

It can be seen that the most important properties of the flow under study are retained in the first terms, in such a way that for the present purposes the series can be truncated at $n = 4$.

3. Normal mode analysis

3.1. Multiscale formulation

The stability study follows a multiscale approach, where we introduce the small parameter $\varepsilon = \dfrac{1}{R}$ and the slow spatial and temporal variables

$$x_1 = \varepsilon x,\ t_1 = \varepsilon t. \tag{8}$$

In the multiscale frame, it can be seen that the same base flow written as an asymptotic expansion can be algebraically rearranged in the form

$$u = \partial_y \Psi = u_0(x_1, y) + \varepsilon u_1(x_1, y) + \cdots \tag{9}$$

$$v = -\partial_x \Psi = -\varepsilon \partial_{x_1} \Psi = \varepsilon v_1(x_1, y) + \cdots \tag{10}$$

what gives also a multiscale expression for the stream function Ψ.

Then, the steady stream function Ψ is perturbed by a small function ψ depending on time, expressed as follows:

$$\psi = \varphi(x, y, t; \varepsilon)\, \mathrm{e}^{i\theta(x,t;\varepsilon)} = [\varphi_0(x_1, y, t_1) + \varepsilon\, \varphi_1(x_1, y, t_1) + \cdots]\ \mathrm{e}^{i\theta(x,t;\varepsilon)}. \tag{11}$$

According to Whitham theory [19], the perturbation has complex wave number h_0 and pulsation σ_0 given by the relations

$$\partial_x\theta = h_0 = k_0 + is_0 \tag{12}$$

$$\partial_t\theta = -\sigma_0 = -(\omega_0 + ir_0). \tag{13}$$

The superposition $\hat{\Psi} = \Psi + \psi$ must satisfy the NS equation for stream functions:

$$\partial_t\nabla^2\hat{\Psi} + \hat{\Psi}_y\partial_x\nabla^2\hat{\Psi} - \hat{\Psi}_x\partial_y\nabla^2\hat{\Psi} = \frac{1}{R}\nabla^4\hat{\Psi}\,. \tag{14}$$

In terms of x_1, t_1 and θ, the spatial and temporal derivatives transform as

$$\partial_x \to h_0\partial_\theta + \varepsilon\partial_{x_1}, \quad \partial_t \to -\sigma_0\partial_\theta + \varepsilon\partial_{t_1}. \tag{15}$$

By applying this transformation to eq. (14) and linearizing, a hierarchy of ordinary differential equations, truncated at the first order in ε, is obtained. The zero order equation is an Orr-Sommerfeld equation, parametric in x_1 and R:

$$\mathcal{A}\varphi_0 = \sigma_0\mathcal{B}\varphi_0 \tag{16}$$

$$\varphi_0 \to 0 \quad \text{as} \quad |y| \to \infty \tag{17}$$

$$\partial_y\varphi_0 \to 0 \quad \text{as} \quad |y| \to \infty\,. \tag{18}$$

where $\mathcal{A} = \left\{(\partial_y^2 - h_0^2)^2 - ih_0R\left[u_0(\partial_y^2 - h_0^2) - u_0''\right]\right\}$, $\mathcal{B} = -iR(\partial_y^2 - h_0^2)$. By numerically solving the zero-order equation and selecting the eigenvalue with the largest imaginary part, a first approximation of the dispersion relation can be obtained, $\sigma_0 = \sigma_0(x_1; h_0, R)$.

The numerical analysis of this relation yields the loci of the branching points where $\partial\sigma_0/\partial h_0 = 0$, (saddle points of the dispersion relation), leading to the determination of wave number and pulsation of the most unstable mode for each x along the wake.

The first-order theory gives the non-homogeneous Orr-Sommerfeld equation, parametric in x_1 and R:

$$\mathcal{A}\varphi_1 = \sigma_0\mathcal{B}\varphi_1 + \mathcal{M}\varphi_0 \tag{19}$$

$$\varphi_1 \to 0 \quad \text{as} \quad |y| \to \infty \tag{20}$$

$$\partial_y\varphi_1 \to 0 \quad \text{as} \quad |y| \to \infty\,. \tag{21}$$

The operator

$$\mathcal{M} = \Big\{ \left[R(2h_0\sigma_0 - 3h_0^2 u_0 - u_0'') + 4ih_0^3\right] \partial_{x_1} + (Ru_0 - 4ih_0)\partial^3_{x_1yy} - Rv_1(\partial_y^3 - h_0^2\partial_y) + Rv_1''\partial_y + ih_0 R\left[u_1(\partial_y^2 - h_0^2) - u_1''\right] + R(\partial_y^2 - h_0^2)\partial_{t_1} \Big\} \tag{22}$$

is a function of the zero-order dispersion relation and eigenfunction as well as of the base flow; it accounts explicitly for the non parallelism of the wake through the presence of transverse velocity v_1.
To avoid secular terms in the solution of (19-21), the non-homogeneous term in equation (19) must be orthogonal to each solution of the adjoint homogenous problem. By rewriting the zero-order eigenfunction in the form $\varphi_0(x_1,t_1,y) = A(x_1,t_1)\zeta_0(x_1,y)$, we introduce a spatio-temporal modulation factor A, and applying the orthogonality condition it is possible to obtain an evolution equation for this function:

$$(\partial_{x_1}A)\int_{-\infty}^{\infty} \zeta_0^+ \left[M_1 + M_2\partial_y^2\right]\zeta_0\,\mathrm{d}y + (\partial_{t_1}A)\int_{-\infty}^{\infty} \zeta_0^+\left[M_7 + M_8\partial_y^2\right]\zeta_0\,\mathrm{d}y + A\int_{-\infty}^{\infty}\zeta_0^+\left[M_1\partial_{x_1} + M_2\partial^3_{x_1yy} + M_3 + M_4\partial_y + M_5\partial_y^2 + M_6\partial_y^3\right]\zeta_0\,\mathrm{d}y = 0, \tag{23}$$

where the coefficients M_j are given in table 1 and ζ_0^+ is the perturbation stream

$M_1 = R(2h_0\sigma_0 - 3h_0^2u_0 - u_0'') + 4ih_0^3$	$M_2 = Ru_0 - 4ih_0$
$M_3 = -ih_0R(\partial_y^2 + h_0^2)u_1$	$M_4 = -R(\partial_y^2 + h_0^2)v_1$
$M_5 = ih_0Ru_1$	$M_6 = Rv_1$
$M_7 = -Rh_0^2$	$M_8 = R$

Table 1.1: Coefficients M_j

function of the homogeneous adjoint problem. If the multiscale analysis is only based on the slow spatial evolution of the system, eq. (23) looses the temporal derivative and the modulation equation is simply $\mathrm{d}_{x_1}A(x_1) = ih_1(x_1)A(x_1)$, where h_1 is a function of $M_i, i = 1,...6$ [17]. By substituting $A(x_1,t_1)$ with $e^{a(x_1,t_1)}$, following Bouthier [4] and going back to the original coordinates x and t, equation (23) can be easily put in the form

$$\partial_t a + p(x)\partial_x a + \varepsilon q(x) = 0 \tag{24}$$

where coefficients $p(x)$ and $q(x)$ are not singular.

The modulation equation can be solved numerically by specifying the initial and boundary conditions. The considered spatial domain extends from a few body-scales D downstream from the body to the far field. Since only one boundary condition has to be imposed, modulation is required to satisfy the asymptotic uniformity in the far field $x = x_f$, that is, the Neumann condition

$$(\partial_x a)_{x=x_f} = 0, \; \forall\, t, \tag{25}$$

while a natural choice for the initial condition is

$$a_{x,t=0} = (const)\,(1+i). \tag{26}$$

Finally, by writing also the first order function in the form $\varphi_1(x_1, y, t_1) = A(x_1, t_1)\zeta_1(x_1, y)$, the complete solution (order 0 + order 1) becomes

$$\psi = (\varphi_0 + \varepsilon\varphi_1)e^{i\theta} = A(\zeta_0 + \varepsilon\zeta_1)e^{i\theta} = (\zeta_0 + \varepsilon\zeta_1)e^{i\theta + a} = (\zeta_0 + \varepsilon\zeta_1)e^{i\theta + i\theta_1}\ , \tag{27}$$

and the complete phase becomes $\Theta = \theta + \theta_1$, where $a = \mathrm{i}\theta_1$. Due to the multiscaling, the wave number is finally $h = \partial\Theta/\partial x = h_0\partial\Theta/\partial\theta + \varepsilon\partial\Theta/\partial x_1 = h_0 + \varepsilon\partial\theta_1/\partial x_1$ and the pulsation is $\sigma = -\partial\Theta/\partial t = -\sigma_0\partial\Theta/\partial\theta + \varepsilon\partial\theta_1/\partial t_1 = -\sigma_0 + \varepsilon\partial\theta_1/\partial t_1$. The first order corrections of the instability characteristics are thus obtained as

$$h_1 = \partial\theta_1/\partial x_1 = k_1 + is_1\ ,\ \sigma_1 = -\partial\theta_1/\partial t_1 = \omega_1 + ir_1\ . \tag{28}$$

3.2. Perturbation hypotheses

Coefficients $p(x)$ and $q(x)$ of the modulation equation (24) are functions of the 0-order perturbation and of the base flow. The base flow is only present in p through the zero-order longitudinal velocity u_0, while the first order longitudinal and transversal velocities u_1 and v_1 are instead present in q. The distributions of the real and imaginary parts of coefficients p and q can be computed by inserting in h_0 and σ_0 the desired values for the 0-order perturbation. Finally, the instability characteristics of the wake including the first order corrections can be computed by numerically solving the modulation equation (24) and using relations (28), and this can be done in different ways, depending on the perturbation hypothesis.

A first physically meaningful choice is to set h_0 and σ_0 as the wave number and pulsation of the most unstable perturbation at a given position x in the wake [18]. An example of the corresponding results for h_0 and σ_0 values relevant to a x position close to the near wake is shown in Fig.1, including comparisons

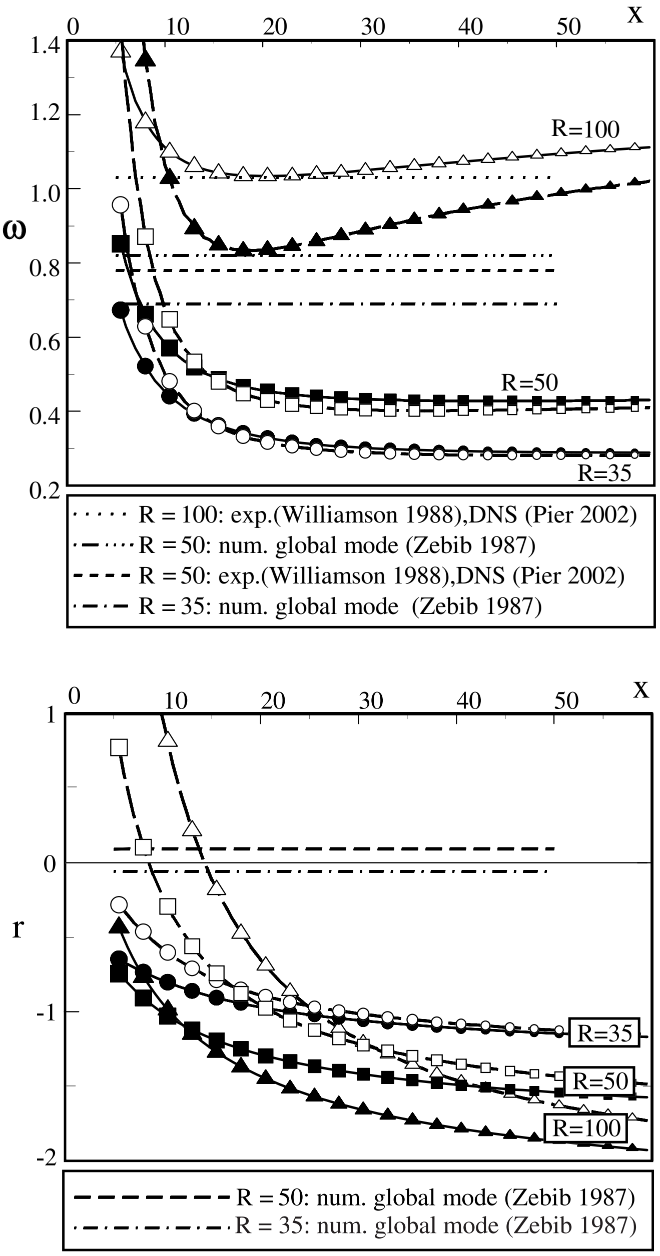

Figure 1: Order 0 (—-, solid symbols) and order 0 + 1 (– –, empty symbols) pulsation and temporal amplification factor x distributions for $R = 35, 50, 100$. Disturbances with complex wave number equal to that of the dominant saddle of the local dispersion relation at x=5.5, that is for $R = 35$, $h_0 = 0.318 - 1.433\,i$, for $R = 50$, $h_0 = 0.451 - 1.913\,i$, for $R = 100$, $h_0 = 1.030 - 2.578\,i$. Comparison with experimental and numerical simulations results [21, 12, 20].

with literature data. Here a large difference between the complete (order 0 + order 1) problem and the order 0 problem is shown more by the pulsation ω and the temporal growth factor r, than by the wave number k and the spatial growth factor s. Typically, the correction increases the values of k and s, thus it reduces the module of the spatial amplification, as also the simpler spatial multiscale analysis shows [17]. The correction for k is higher as it approaches the body and R is increased. The correction for s is negligible throughout, regardless of R, except for $R = 50$ and 100 at the beginning of the domain here considered.

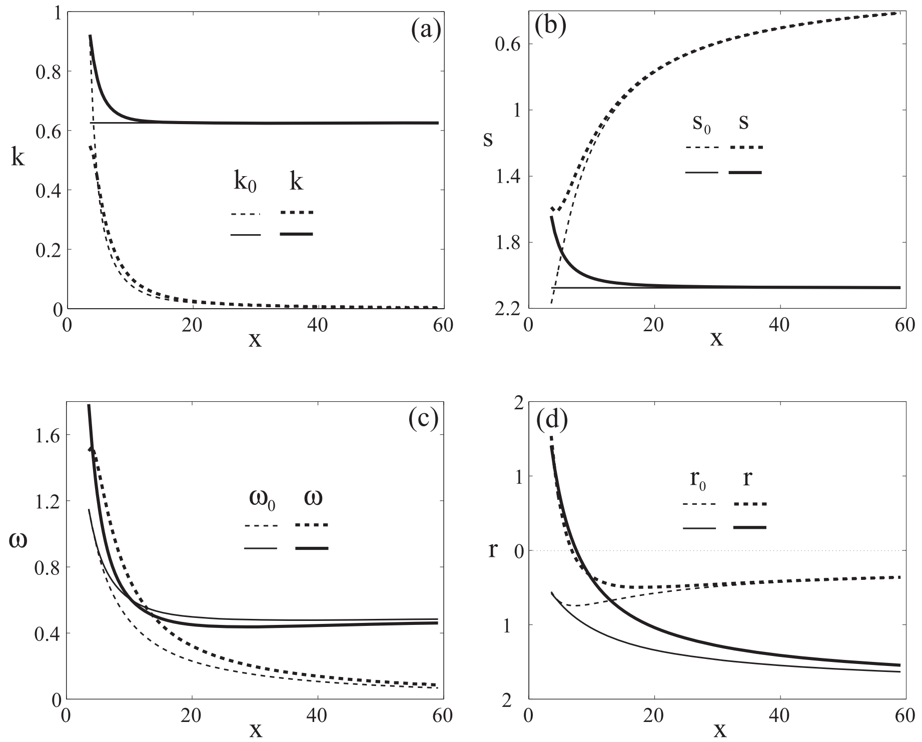

Figure 2: Comparison between the instability characteristics with the ‘saddle points sequence”’ perturbation hypothesis (dashed lines) and the perturbation with $h_0(x = 4.10)$=const. at R=50 (solid lines): (a) wave number, (b) spatial growth rate, (c) pulsation, (d) temporal growth rate.

Another natural choice is to set h_0 and σ_0 as the *local* wave number and pulsation of the most unstable perturbation for each x in the wake [2] (saddle point sequence). So doing, the disturbance is locally tuned, through the modulation function, to the property of the instability as can be seen from the zero-order

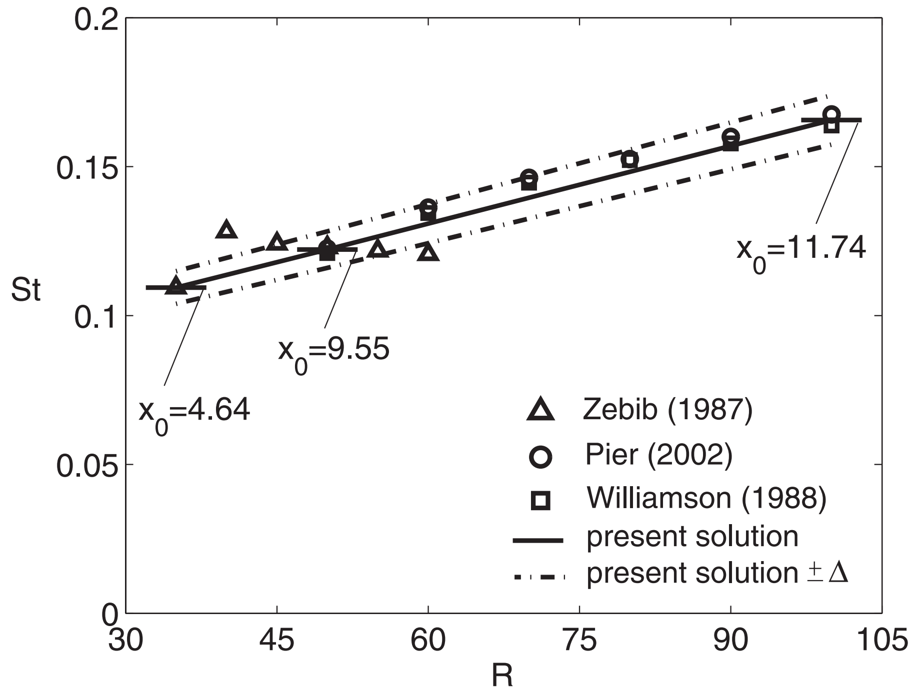

Figure 3: Comparison between the pulsation (in terms of Strouhal number) data according to Pier (2002), Zebib (1987), Williamson (1988) and present solution (accuracy $\Delta = 0.05$).

theory (near-parallel parametric Orr-Sommerfeld treatment). This leads to a synthetic analysis of the nonparallel correction on the instability characteristics. In such a way, the evolution of the zero order dispersion relation is directly inserted into the variable coefficients of the modulation equation. The streamwise variation of the instability characteristics is deduced from the spatial and temporal derivatives of the modulation function. With this new approach, the system is considered as locally perturbed by waves with a wave number that varies along the wake and which is equal to the wave number of the dominant saddle point of the zero order dispersion relation, taken at different Reynolds numbers. Since the perturbation is no more parameterized with respect to a given wave number, the Reynolds number remains the only parameter of the present stability analysis. An example of the corresponding results, compared with the ones parameterized by the most unstable mode of a position in the near wake, is shown in Fig. 2. One can see that the two perturbation hypothesis yield the same results close to $x = 4.10$, but differ downstream, where, in this case, the perturbation turns out finally in a greater response from the flow.

Within the saddle point perturbative hypothesis, results concerning the Strouhal number are compared, in Fig. 3, with data from the global results obtained by Pier [12] (DNS simulations), Williamson [20] (laboratory observations) and Zebib [21] (numerical experiments). In this figure, the x positions pointed out represent the wake section where the longitudinal distribution of pulsation, obtained with the present method, match the global pulsation obtained in these numerical and laboratory experiments. A linear interpolation on the points we have determined shows that the experimental data fall within an accuracy of $\pm 5\%$ around a pulsation curve that grows with the Reynolds number.

4. Initial-value problem

A general three-dimensional initial-value perturbation problem is here presented to study the linear stability of a two-dimensional growing wake. The base flow has been obtained by approximating it with the expansion solution for the longitudinal velocity component that considers the lateral entrainment process (see §2 and [16] for details). By imposing arbitrary three-dimensional perturbations in terms of the vorticity, the temporal behaviour, including both the early time transient as well as the long time asymptotics, is considered [3], [9], [8]. The approach has been to first perform a Laplace-Fourier transform of the governing viscous disturbance equations and then resolve them numerically by the method of lines. The base model is combined with a change of coordinate [5], [6]. Base flow configurations corresponding to a R of 50, 100 and various physical inputs are examined.

4.1. Formulation

The base flow is viscous and incompressible. To define it, the longitudinal component of the Navier-Stokes expansion for the two-dimensional steady bluff body wake presented in §2 (see [16], [1]) has been used. The x coordinate is parallel to the free stream velocity, the y coordinate is normal. The coordinate x_0 plays the role of parameter of the system together with the Reynolds number. By changing x_0, the base flow profile locally approximates the behaviour of the actual wake generated by the body. The equations are

$$\nabla^2 \widetilde{v} = \widetilde{\Gamma} \tag{29}$$

$$\frac{\partial \widetilde{\Gamma}}{\partial t} + U\frac{\partial \widetilde{\Gamma}}{\partial x} - \frac{\partial \widetilde{v}}{\partial x}\frac{d^2U}{dy^2} = \frac{1}{R}\nabla^2\widetilde{\Gamma} \tag{30}$$

$$\frac{\partial \widetilde{\omega}_y}{\partial t} + U\frac{\partial \widetilde{\omega}_y}{\partial x} + \frac{\partial \widetilde{v}}{\partial z}\frac{dU}{dy} = \frac{1}{R}\nabla^2\widetilde{\omega}_y \tag{31}$$

where $\widetilde{\omega}_y$ is the transversal component of the perturbation vorticity, while $\widetilde{\Gamma}$ is defined as $\widetilde{\Gamma} = \frac{\partial \widetilde{\omega}_z}{\partial x} - \frac{\partial \widetilde{\omega}_x}{\partial z}$. All physical quantities are normalized with respect to the free stream velocity, the spatial scale of the flow D and the density. Equation (29) is a kinematic identity. Equations (30) and (31) are the Orr-Sommerfeld and Squire equations respectively, known from the classical linear stability analysis for three-dimensional disturbances. By performing a combined Laplace-Fourier decomposition of the dependent variables in terms of x and z, the governing equations become

$$\frac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{v} = \hat{\Gamma} \tag{32}$$

$$\begin{aligned}\frac{\partial \hat{\Gamma}}{\partial t} &= -(ikcos(\phi) - \alpha_i)U\hat{\Gamma} + (ikcos(\phi) - \alpha_i)\frac{d^2U}{dy^2}\hat{v} \\ &+ \frac{1}{R}[\frac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\Gamma}]\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{\partial \hat{\omega}_y}{\partial t} &= -(ikcos(\phi) - \alpha_i)U\hat{\omega}_y - iksin(\phi)\frac{dU}{dy}\hat{v} \\ &+ \frac{1}{R}[\frac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\omega}_y]\end{aligned} \tag{34}$$

where $\hat{g}(y,t;\alpha,\gamma) = \int_{-\infty}^{+\infty}\int_{0}^{+\infty} \widetilde{g}(x,y,z,t)e^{-i\alpha x - i\gamma z}dxdz$ is the combined Laplace-Fourier transform of a general dependent variable, $\phi = tan^{-1}(\gamma/\alpha_r)$ is the angle of obliquity, $k = \sqrt{\alpha_r^2 + \gamma^2}$ is the polar wavenumber and $\alpha_r = kcos(\phi)$, $\gamma = ksin(\phi)$ are the wavenumbers in x and z directions respectively. In order to have a finite perturbation kinetic energy, the imaginary part α_i of the complex longitudinal wavenumber can only assume non-negative values.

Perturbative equations (29), (30) and (31) are to be solved subject to appropriate initial and boundary conditions. As the governing equations are expressed in term of vorticity, arbitrary initial conditions are given for $\widetilde{\Gamma}$ and $\widetilde{\omega}_y$ in the physical coordinates x, y, z. Among all solutions, we seek those whose velocity field is zero in the free stream.
The partial differential equations (32)-(34) are numerically solved by method of lines. The spatial derivatives are centre differenced and the resulting system is then integrated in time by an adaptative multi-step method (variable order Adams-Bashforth-Moulton PECE solver).

The initial conditions can be shaped in terms of set of functions in the L^2 Hilbert space. We choose

$$\hat{v}(0,y) = e^{-(y-y_0)^2} cos(n_0(y-y_0)) \quad \text{and} \quad \hat{v}(0,y) = e^{-(y-y_0)^2} sin(n_0(y-y_0)),$$

for the symmetric and the asymmetric perturbations, respectively. We recall that the trigonometrical system is a Schauder basis in each space $L^p[0,1]$, for $1 < p < \infty$. More specifically, the system $(1, sin(n_0 x), cos(n_0 y), \dots)$, where $n_0 = 1, 2, \dots$, is a Schauder basis for the space of square-integrable periodic functions with period 2π. This means that any element of the space L^2, where the dependent variables are defined, can be written as an infinite linear combination of the elements of the basis.
Parameter n_0 is the oscillatory parameter for the shape function, while y_0 is a parameter which controls the distribution of the perturbation along y (by moving away or bringing nearer the perturbation maxima from the axis of the wake). It should be noticed, that by (32), the initial $\hat{\Gamma}$ is not zero at finite y.

The vorticity field is immediately generated from the interaction of the initial three-dimensional perturbation field and the mean vorticity of the base flow. It can be verified that the eventual introduction of an initial transversal vorticity perturbation $\hat{\omega}_y(0,y) \neq 0$ does not substantially modify the results in the transient and long terms, thus we put $\hat{\omega}_y(0,y) = 0$.

4.2. Early transient and asymptotic behaviour of perturbations

The effects of various initial conditions and subsequent transient behaviour is one of the main aspects of the present study. The evolution of disturbances is characterized by the kinetic energy density

$$\begin{aligned} e(t;\alpha,\gamma) &= \frac{1}{2}\frac{1}{2y_d}\int_{-y_d}^{+y_d} (|\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2) dy \\ &= \frac{1}{2}\frac{1}{2y_d}\frac{1}{|\alpha^2+\gamma^2|}\int_{-y_d}^{+y_d} (|\frac{\partial \hat{v}}{\partial y}|^2 + |\alpha^2+\gamma^2||\hat{v}|^2 + |\hat{\omega}_y|^2) dy, \quad (35) \end{aligned}$$

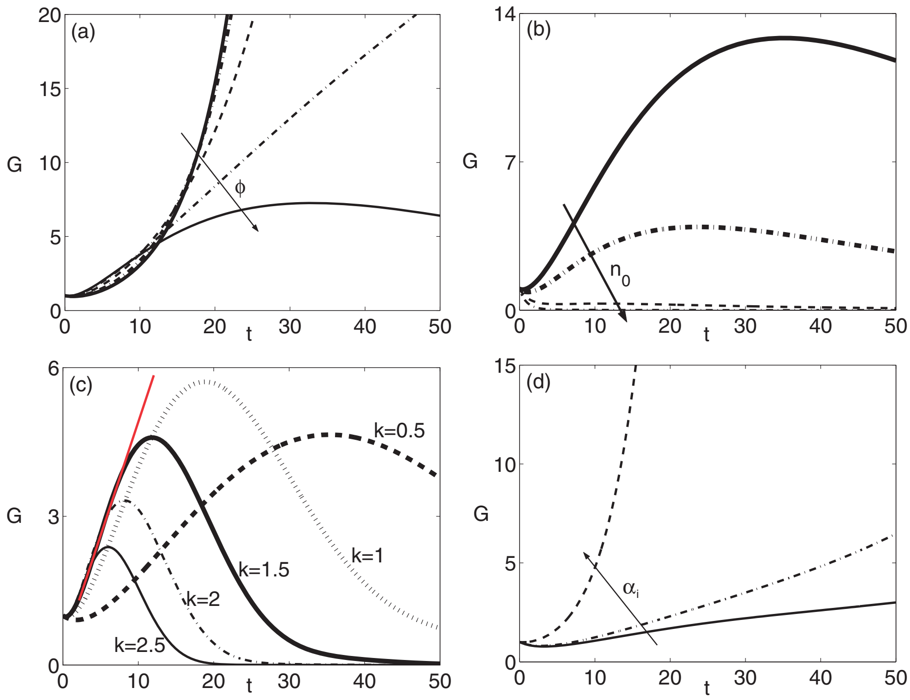

Figure 4: The amplification factor G as function of time. (a) $R = 100$, $k = 1.5$, $\alpha_i = 0.01$, $n_0 = 1$, $y_0 = 0$, $x_0 = 14.00$, symmetric initial condition, $\phi = 0, \pi/8, \pi/4, (3/8)\pi, \pi/2$. (b) $R = 100$, $k = 1.5$, $\alpha_i = 0.015$, $\phi = \pi/2$, $y_0 = 0$, $x_0 = 8.00$, asymmetric initial condition, $n_0 = 1, 3, 5, 7$. (c) $R = 100$, $\alpha_i = 0.01$, $n_0 = 1$, $\phi = (3/8)\pi$, $y_0 = 0$, $x_0 = 6.50$, asymmetric initial condition, $k = 0.5, 1, 1.5, 2, 2.5$. (d) $R = 50$, $k = 1.2$, $n_0 = 1$, $\phi = \pi/4$, $y_0 = 0$, $x_0 = 9.50$, symmetric initial condition, $\alpha_i = 0, 0.01, 0.1$.

where $2y_d$ is the extension of the spatial numerical domain. The value y_d is defined so that the numerical solutions are insensitive to further extensions of the computational domain size. Here, we take $y_d = 15$. The total kinetic energy can be obtained integrating the energy density over all α and γ. We introduce the normalized amplification factor $G(t)$

$$G(t; \alpha, \gamma) = \frac{e(t; \alpha, \gamma)}{e(t = 0; \alpha, \gamma)}, \tag{36}$$

that can effectively measure the growth of a disturbance of wavenumbers (α, γ) at the time t, for a given initial condition at $t = 0$ (see [8], [9]).
The temporal growth rate r is defined as

$$r(t; \alpha, \gamma) = \frac{log|e(t; \alpha, \gamma)|}{2t}, \tag{37}$$

and is introduced to evaluate both the early transient as well as the asymptotic behaviour of the perturbation. It is noticed that, by definition, r is singular for $t = 0$. Quantity r has in fact a precise physical meaning as an asymptotic property of the perturbation. The angular frequency (pulsation) f of the perturbation is defined by considering the phase φ of the complex wave at a fixed transversal station (for example $y = 1$) and then computing its temporal derivative

$$f(t;\alpha,\gamma) = \frac{d\varphi(t;\alpha,\gamma)}{dt}. \tag{38}$$

In the following, we present a summary of the significant behaviour shown during the transient by the three-dimensional perturbations (Fig. 4). Case (a) shows that a growing wave becomes damped, increasing the obliquity angle beyond $3/8\pi$. Case (b) shows that the damping is more rapid for higher values of n_0, that is when the perturbation oscillates many times across the basic flow. Case (c) demonstrates that perturbations almost normal to the base flow ($\phi = (3/8)\pi$) are all asymptotically stable when changing their polar wavenumber. It can be observed that increasing k ($k \to \infty$) the growth rate of the transient seems to tend to a limiting value (see the grey line in the figure 4). The maximum growth is reached for $k \sim 1$. In the end, in case (d) an interesting phenomenon can also be observed. The perturbation parameters in this case corresponds to values far from the saddle point of the Orr-Sommerfeld dispersion relation at the section considered. It can be seen that, by increasing the order of magnitude of α_i, perturbations that are more rapidly damped in space lead to a faster growth in time.

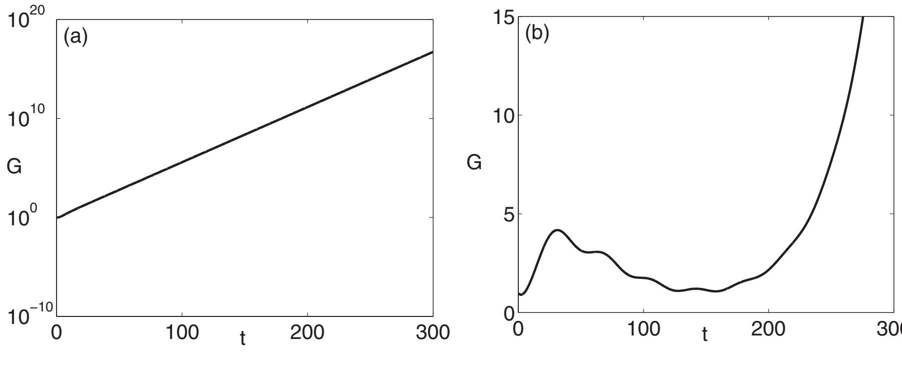

Figure 5: The amplification factor G as function of time. Effect of the symmetry of the disturbance. $R = 100$, $k = 0.6$, $\alpha_i = 0.02$, $n_0 = 1$, $y_0 = 0$, $x_0 = 11$, $\phi = \pi/4$. (a) symmetric initial condition, (b) asymmetric initial condition.

In Fig. 5 the influence of the perturbation symmetry on the early time behaviour is shown. Both the configurations are amplified asymptotically (symmetric case (a): r = 0.0614, asymmetric case (b): r = 0.0038), but transients are very different. In the symmetric case (a) the growth is immediate and monotone and the perturbation quickly reaches the asymptotic state (note that in part (a) of this figure a logarithmic scale is used on the ordinate). The asymmetric case (b) instead presents a particular temporal evolution. After a maximum and a minimum of energy are reached, the perturbation is slowly amplifying and the transient can be considered extinguished only after hundreds time scales. This particular configuration shows a behaviour that is generally observed in this analysis, that is, asymmetric conditions lead to transient evolutions that last longer than the corresponding symmetric ones.

5. Final Comments

A first conclusion we can draw from this linear stability study is a tribute to the scientific quality of the Orr-Sommerfeld modal analysis that is very synthetic and prove to be a powerful mean to obtain the asymptotic stability state of a flow system. The relevant perturbative equations contains in their coefficients the description of the steady configuration whose stability is considered. Under the condition that this motion is physically described in a rigorous way, which necessarily should include all the phenomena that produce its spatial evolution, the Orr-Sommerfeld theory yields results in agreement with experimental findings. Recently this was done in two steps: - first, by looking for a fully nonlinear Navier-Stokes asymptotic expansion solution of the basic flow, - second, by introducing a spatio-temporal multiscaling. The multi-scaling produces in the perturbative equation a term which represents explicitly the transversal dynamics responsible for the entrainment and the spatial growth of the system. A second recent improvement in the theory is due to the fact that to first order in the multi-scaling and moving downstream towards the far field, it is possible to deduce the sequence of the most unstable or less stable points in the dispersion relationship. This is paramount to obtain a local dispersion relationship depending solely on the Reynolds number and the longitudinal position in the flow. The initial value analysis considers three-dimensional instability waves and determines their temporal and transversal linear evolution. It can be found that if the longitudinal wave is represented by a wave that spatially decays ($\alpha_i > 0$) or remains constant ($\alpha_i = 0$), the asymptotic fate of the perturbation is the same deduced by means of the modal theory. An important point highlighted by the initial value problem is that the early and intermediate transients can be of a great variety, not a priori intuitable. Waves may continuously grow or decay or may initially grow and subsequently decay, which may lead to a form of transition called bypass transition. Not only, waves may transit across various phases, an initial growth, followed by a slow decay and then by a second growth. The time scales of these transients may last hundred basic flow time scales. This knowledge is important because it opens a question on the interpretation of the flow properties obtained from numerical flow simulations. At the state of the art, the numerical simulations typically produce temporal evolutions lasting a few basic time scales only, which could not be a sufficient observation window to study the transient behaviour.

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RELAZIONE DELLA COMMISSIONE

sulla memoria *Hydrodynamics linear stability theory. A comparison between Orr-Sommerfeld modal and initial value problem analyses* di DANIELA TORDELLA, STEFANIA SCARSOGLIO e MARCO BELAN presentata dal Socio MICHELE ONORATO nell'adunanza del 13 giugno 2007 e approvata nell'adunanza del 21 novembre 2007

Il lavoro presentato discute due formulazioni teoriche della teoria della stabilità applicate ad un problema classico della dinamica dei fluidi, la scia bidimensionale di un corpo tozzo, che tuttavia ha svariati legami con problemi concreti nell'ambito dell'ingegneria del vento e dei mezzi di trasporto.

La prima teoria discussa è la teoria modale di Orr-Sommerfeld, presentata per la prima volta esattamente 100 anni fa alla Royal Irish Academy of Science. La seconda è la teoria non modale fondata sulle equazioni perturbative in stazionarie di Squires e di Orr-Sommerfeld, ed è matematicamente lo studio del problema ai valori iniziali di perturbazioni tridimensionali rappresentate nello spazio di Fourier associato al piano che contiene la direzione longitudinale e quella traversale al flusso base di cui si studia la stabilità.

Nel primo caso la teoria modale è discussa in modo molto attuale tenendo conto del non parallelismo del flusso base, inserendo cioè esplicitamente nell'equazione di Orr-Sommerfeld la completa dinamica trasversale del flusso base che è non lineare. Questo è stato fatto introducendo una procedura di multiscaling spaziale e temporale che ha condotto ad un problema di stabilità non omogeneo. Si sono ottenuti degli ottimi risultati che offrono un quadro di instabilità assoluta in regioni del flusso compatibili con le premesse teoriche di flusso sottile. La contraddizione WKBJ è stata superata e le frequenze tipiche ottenute per le onde di instabilità risultano in ottimo accordo con i risultati sperimentali. Inoltre una forma molto sintetica di relazione di dispersione – dipendente solo dal numero di Reynolds e dalla posizione nella scia – è stata proposta perturbando il sistema con i dati associati ai punti di sella della relazione di dispersione, ottenuta all' ordine zero nel multiscaling.

Nella seconda parte del lavoro si discute l'evoluzione temporale della densità di energia di perturbazioni tridimensionali rappresentate nella formulazione mista velocità–vorticità. Queste soluzioni lineari non modali possono avere crescite sia esponenziali sia algebriche e tengono conto dell'inclinazione della perturbazione rispetto al flusso base. Un quadro ampio di possibili transitori è presentato. Si dimostra che il loro stato asintotico si adagia sempre sui risultati dati dalla teoria modale.

In base all'esame del lavoro, la Commissione unanime ne raccomanda la pubblicazione all'Accademia delle Scienze.

La Commissione ritiene poi che l'impostazione data al lavoro, la completezza dell'analisi condotta e i risultati ottenuti, caratterizzino il lavoro più come memoria che come nota. Se ne raccomanda quindi la pubblicazione in forma di Memoria.

Torino, 26 ottobre 2007

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