On the angular momentum, Helmholtz and higher order vorticity equations. Application to turbulent flows.

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Abstract

We consider the equation for the intrinsic moment of momentum averaged over small volumes of linear dimension $\delta$. We give a representation of it as an infinite sequence of independent equations by means of a series expansion in terms of $\delta^2$. The equations of different order are obtained through linear antisymmetric operators - with a structure similar to that of the curl - acting on the momentum equation. The first order term of the sequence is just the Helmholtz equation, the remaining terms can be viewed as a balance for a kind of higher orders vorticity.

We show that the coupling between the momentum and the angular momentum equation, based on a supposed antisymmetric part of the stress tensor - which has been sometimes assumed by authors dealing with turbulent flow of homogeneous fluid - is devoid of physical rationale. We propose a different form of coupling which may be used in describing turbulent flow of an homogeneous medium, by means of a large eddy simulation technique. In our model the coupling is given by a functional dependence of the turbulent eddy diffusivity over the angular momentum of a finite volume of fluid.

1. Introduction

To deal with a non-homogeneous fluid with internal structures on which external forces and couples act, it is necessary to introduce a balance of angular momentum, whose presence affects the symmetry property of the stress tensor. In this case the moment of the momentum equation is no more equivalent to the angular momentum budget and a new variable, the intrinsic angular momentum per unit volume, must necessarily be brought in.

The micropolar theory views the medium as a collection of material systems, the micro-elements, owning momentum, intrinsic angular momentum and energy. The micro-elements may contain internal structures (like liquid crystals, blood cells,...), however the fluid is viewed as monophase. The motion of the microelement is fully described by the velocity of its centroid and by a second order tensor - called by Eringen microgyration tensor - which portrays the internal deformation and rotation of the element. The centroid velocity and the microgyration tensor vary continuously in the external scale of the field. In
an incompressible flow Eringen’s theory leads to a set of twelve differential equations, from which the intrinsic moment of momentum can be recovered by taking the antisymmetric part of the microgyration tensor. The equation of the intrinsic angular momentum appears to be coupled to the momentum equation expressed in terms of the centroid velocity in such a way as to establish a link between the intrinsic motion inside the microelements and the mean velocity field.

During the last century the equation of angular momentum balance has been applied few times to discuss the behavior of turbulent flows. Since the earliest Mattioli’s application in 1933, the equations of motion were often integrated over finite volumes to evidence the evolution of the large scales of the turbulence. Even with fluids deprived of internal structures, asymmetry was associated to the turbulent stress in Mattioli\cite{9,10} and in Nicolaevsky.\cite{11,12} In Eringen\cite{13} the micromorphic theory was applied to turbulence.

In Sec. 2 we carried out an analysis about the structure of the angular momentum budget over finite volumes of linear dimension $\delta$. The analysis is relevant to all situations where the fluid may be considered locally homogeneous. Through a power series development in the square of the linear dimension of average we show that the balance for the intrinsic momentum may be represented by an infinite succession of independent equations obtained applying linear antisymmetric operators to the momentum balance. The first order term of the sequence is the vorticity equation,\cite{15,16} while the higher order relations are not reducible to it and may be viewed as high order vorticity budgets.

Applications to turbulence of theories relevant to structured flows are discussed in Sec. 3.1 through the analysis of the symmetry property of the Navier-Stokes equations. We discuss the common aspects of these theories and their physical support. In Sec. 3.2, in the ambit of turbulence modeling, we suggest a different kind of coupling between the momentum and angular momentum turbulent equations, which does not rely on a supposed existence of the antisymmetric part of the stress tensor. We propose a large eddy scale model based on the proportionality of the turbulent diffusivity to the intrinsic moment of momentum. The principal features of this model are the correct scaling of the eddy diffusivity $\nu_\delta$ with respect to both the filtering length $\delta$ and the dissipation rate function and the introduction of a differential equation - the intrinsic angular momentum equation - to follow the evolution of $\nu_\delta$. Thing that should be convenient in case of simulation of non-equilibrium turbulence fields. A natural application for this model would be the turbulent motion of suspensions of massive particles on which external couples apply, such as is for example a dusty plasma flow. In such a case the model would not be anymore differential since the coupling between the momentum and angular momentum equations is already present and associated to the physics of the problem.

2. Angular momentum and vorticity in monophase flows

We consider the relationship between angular momentum and vorticity and between their equations. It is usually assumed that vorticity represents the angular velocity of a small volume of fluid. Its equation is obtained by taking the curl of the momentum equation. Working on volume averaged equations, at a first order approximation of the angular momentum expanded in the square of the linear dimension of average $\delta$, Nigmatulin and Nikolaevsky\cite{15} and Chatwin\cite{16} showed the proportionality of vorticity to the angular momentum. The fluid was taken as incompressible.

Here we expand the intrinsic angular momentum, defined as the angular momentum
about the center of mass of the fluid element, by means of a power series in $\delta^2$. The expansion is carried out at the general order $m$. As a result an infinite sequence of independent linear antisymmetric differential operators, the first of which is the curl, is obtained. Applying the operators to the momentum balance a correspondent infinite sequence of independent equations is also obtained, the first order term is just the Helmholtz equation.

We introduce as spatial average in the neighborhood of a point $x$

$$\mathcal{I}_\delta = \{ x + \eta \in \mathbb{R}^3 : \| \eta \| < \delta \},$$

the operator $< \cdot >_\delta$

$$< \varphi >_\delta (x, t) = \frac{1}{V_\delta} \int_{\mathcal{I}_\delta} \varphi(x + \eta, t) d\eta,$$

where $V_\delta$ is the volume of $\mathcal{I}_\delta$.

The intrinsic moment operator $M$ acting on a vector field $f$ is defined as

$$(Mf)_i = \varepsilon_{ilk}(< x_i f_k > - < x_l > < f_k >)$$

Expanding $M$ in a power series of $\delta^2$ and using the symmetry of $\mathcal{I}_\delta$ we get:

$$M = \sum_{m=0}^{\infty} \frac{\delta^{2m+2}}{(2m+2)!} A^{(m)},$$

where

$$A^{(m)}_{ik} = \varepsilon_{ilk} \sum_{t=0}^{m} \sum_{j=0}^{t} \left( \frac{2m+1}{2t+1} \right) \left( \frac{2t+1}{2j+1} \right) a_{m-t,t-j+1}^t \partial_{p}^{2j+1} \partial_{q}^{2(t-j)} \partial_{k}^{2(m-t)}$$

$$a_{\alpha,\beta,\gamma} = \frac{1}{V_1} \int_{\mathcal{I}_1} \zeta_1^{2\alpha} \zeta_2^{2\beta} \zeta_3^{2\gamma} d\zeta.$$

Here index $p$ is the integer remainder of $(l+1)/3$, while index $q$ is the integer remainder of $(l+2)/3$ (so that $l, p, q$ is a permutation of $1,2,3$). The mean intrinsic angular momentum per unit mass $h$ of each element $\mathcal{I}_\delta$ is defined as $< \rho > h = M(\rho u)$. From (3) we get:

$$h_i = \frac{1}{2} a_{0,0,1} \varepsilon_{ilk} \frac{\partial_l (\rho u_k)}{\rho} \delta^2 + \frac{1}{\rho} \left\{ -\frac{1}{4} a_{0,0,1} \varepsilon_{ilk} \partial_l (\rho u_k) \frac{\nabla^2 \rho}{\rho} + \frac{1}{4!} \left[ 3 a_{0,1,1} \nabla^2 \varepsilon_{ilk} \partial_l (\rho u_k) + (a_{0,0,2} - 3 a_{0,1,1}) \varepsilon_{ilk} \partial_l^3 (\rho u_k) \right] \right\} \delta^4 + O(\delta^6)$$

The operator $M$ applied to the momentum equation

$$f_k(x, t) \equiv \partial_l (\rho u_k) + \partial_j (\rho u_k u_j) - \partial_j \sigma_{kj} - \rho b_k = 0$$

where $\sigma$ is the stress tensor and $b$ is an external force field, yields the budget for $h$ in terms of a series expansion in $\delta^2$. Since this equation is an identity in $\delta$, the coefficient of each $\delta^{2(m+1)}$ must vanish independently; that is $A^{(m)} \mathbf{f} = 0$, $\forall m \in \mathbb{N}$, that reduces to

$$\varepsilon_{ilk} \partial_{l}^{2m+1} f_k(x, t) = 0 \quad \forall m \in \mathbb{N}, \quad i, l, k = 1, 2, 3$$

as it can be shown by induction. Free of any approximation the angular momentum budget leads to a sequence of differential equations that are not reducible one to the other, the
leading order of which is the vorticity equation. Thus \( h \) cannot be just described in terms of the vorticity \( \omega \) only. Only when \( \delta \to 0 \) \( h \) contains the same amount of information as \( \omega \), but in this case it goes to zero as \( \delta^2 \).

The higher order equations of the sequence may be interpreted as balances for high order vorticities. The result is quite general because it is not restricted by the particular nature of the constitutive equation as long as this last describes a fluid - even if with internal structures - as a fluid with bulk properties. The higher order terms of this sequence may turn useful in turbulence applications where auxiliary independent equations are needed for the correlation variables introduced by the filtering process.

3. Angular momentum and turbulence

3.1 Applications to turbulence of structured fluid theories

In Eringen\textsuperscript{13} the microfluid theory, conceived for structured fluid, was applied to a turbulent flow of a non-structured fluid. Analog approaches were proposed in times past by Mattioli,\textsuperscript{9,10} Ferrari,\textsuperscript{14} where an intrinsic angular momentum was introduced to represent the turbulent transport, and by Nikolaevsky,\textsuperscript{11,12} who approximated the volume average of the spatial derivatives in terms of an incremental ratio of surface integrals in such a way introducing the asymmetry into the turbulent stress.

The crucial point of these theories is the coupling between the momentum and the moment of momentum equations. In all of them the distribution of the mean velocities depends upon the motion of internal rotation, considered as the structural property of the elemental cells, the so-called micro-elements. The mathematical coupling between the two kinematical aspects is due to the presence of the antisymmetric part of the stress tensor in both the equations of momentum and angular momentum.

This aspect is explicitly declared in Mattioli, it has been renewed by Ferrari and Nikolaevsky, but it is also a necessary element in the model by Eringen. All these theories seem capable to reproduce experimental results about turbulent sheared flows. In spite of this, their common and decisive component - the coupling between the momentum and angular momentum equations through an antisymmetric part of the stress tensor - is an arbitrary choice, whose validity in the case of homogeneous fluid may be proved false.

The volume average for a function \( f \), already introduced in Sec. 2, may be written as

\[
<f>(x) = \int_{\mathbb{R}^3} g_\delta(x-y)f(y)\,dy = \int_{\mathbb{R}^3} g_\delta(y)f(x-y)\,dy
\]

where \( f \) and \( g_\delta \in L^1 \), together with their derivatives. Function \( g_\delta \) is the weight function shaping the space portion where the average is taken. Under these assumptions the first spatial derivative and the volume average commute:

\[
<\partial_i f>(x) = \int_{\mathbb{R}^3} \partial_i (g_\delta(y)f(x-y))\,dy = \partial_i <f>
\]

Relation (7) implies that this spatial filtering is unable to break the symmetry property of flow tensors, independently from the kind of regime of motion, either laminar or turbulent. To verify this inference it is sufficient to apply the filtering to the Navier-Stokes equations that, in the case of incompressible flow, yields

\[
\rho D_t <u_i> = \partial_j (-<p\delta_{ij}> + \nu(\partial_j <u_i> + \partial_i <u_j>)) + <u_i><u_j> - <u_i u_j>.
\]
The theories by Mattioli and Nikolaevsky adopt the following structure for turbulence equations of a homogeneous fluid:

\[
D_t <u_i> = \partial_j \sigma_{ij} + \partial_j \tau_{ij}^a + \partial_j \tau_{ij}^s \quad (9)
\]

\[
D_t <h_i> = \varepsilon_{ijk} \tau_{jk}^a + \partial_j c_{ij} + \partial_j \varsigma_{ij} \quad (10)
\]

where suffices \(a\) and \(s\) stand for antisymmetric and symmetric and all the tensors are volume averaged quantities: \(\sigma_{ij}\) and \(\tau_{ij}\) are the molecular and turbulent stress tensors, while \(\varsigma_{ij}, c_{ij}\) are respectively the molecular and turbulent flow tensors of angular momentum. There is no doubt that the averaged velocity indicates the same variable in equation (8) and (9) and thus that the equations cannot be both true.

In Mattioli’s theory the antisymmetric part of the turbulent stress is assumed and interpreted as the momentum transport due to the vortical structures of the small scales filtered out from the equation. A model is then needed for this term. He also assumed, not quite legitimately since he dedicated an evolutive equation to it, that the intrinsic moment be proportional to the vorticity. The angular momentum budget becomes thus an equation operating on the vorticity. It has however a different structure than the original Helmoltz equation because of the presence of the term \(\varepsilon_{ijk} \tau_{jk}^a\). So doing one dependent variable is dropped out. The balance is then used to model the turbulent transport coefficient. Contrarily of what done for the momentum, the inertial tensor \(c_{ij}\) is assumed symmetric.

Nikolaevskij\(^1\) while computing (6) over cubes introduces an approximation of the second order in \(\delta\) that induces the lost of the property (7) and thus of the property of symmetry of the averaged equation, where he obtains the divergence of asymmetric tensors. In fact he uses the Gauss theorem to transform the integral of the divergence in a surface integral. Then, he approximates derivatives with the incremental ratios:

\[
< \frac{\partial f}{\partial x_i} > = \frac{\partial}{\partial x_i} [f]_{(i)} + O(\delta^2),
\]

where one must not sum up over the index in parenthesis and \([f]_{(i)}\) is defined by

\[
[f]_{(i)} = (2\delta)^{-2} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} f(x + \eta_j e_j + \eta_k e_k) \, d\eta_j \, d\eta_k.
\]

Nikolaevskij neglects the terms \(O(\delta^2)\). In so doing, together with the commutability, he loosens the symmetry of the tensors involved in the equations.

In the application of the microfluid theory to turbulence, Eringen,\(^1\) the turbulent flow is considered the motion of a ”simple microfluid”, even if any physical internal structure potentially causing asymmetry is missing.

The motion of the micropolar element is described by the mean velocity \(v_k(x, t)\) and by the microgyration tensor \(\nu_{kl}(x, t), (k, l = 1, 2, 3)\). This last arises from the motion and deformations of material points inside the volume of the micro-element. The resulting system of equations - not reducible to the filtered Navier-Stokes equations - comprehends twelve scalar equations for the three components of the mean velocity and for the nine components of \(\nu_{kl}(x, t)\) and contains 23 constant viscosity coefficients. The intrinsic moment of momentum equation, obtainable by taking the antisymmetric part of \(\nu_{kl}(x, t)\), is coupled to the momentum equation through the antisymmetric part of the stress tensor as in Mattioli and Nikolaevsky. In his solution for the 2D turbulent channel flow Eringen
gives a solution of his system of equations where the stress tensor is non symmetric. The constant coefficients, which are only five thanks to the simple domain geometry, were adjusted according to the experimental observations by Laufer.\(^\text{17}\) However it is easily seen that if the non symmetric part of the stress tensor is put equal to zero, the equations result uncoupled and the mean motion would be independent from the internal motion of the micro-elements.

### 3.2 Angular momentum large eddy model for turbulent flows

In this paragraph we would like to put forward a different kind of coupling of the momentum and angular momentum equations, that does not require a non-symmetry part of the stress tensor to exist. In the framework of the large eddy scale simulation, we propose a new differential model for the turbulent stresses based on a Boussinesq transport coefficient proportional to the mean intrinsic moment modulus \(h\), a quantity that is supposed to carry both the mechanism of stretching and the process of auto-diffusion. The coupling between the momentum and moment of momentum equations is thus given by the functional dependence of the eddy diffusivity over the intrinsic angular momentum of a finite volume of fluid. The averaged motion equations are written in the following form:

\[
\begin{align*}
D_t <u_i> &= \partial_j \sigma_{ij} + \partial_j \tau_{ij} + b_i \quad \text{(11)} \\
D_t h_i &= \partial_j \varsigma_{ij} + \partial_j c_{ij} + \beta_i \quad \text{(12)}
\end{align*}
\]

where \(h_i\) is the intrinsic angular momentum, see Sec. 2, \(\sigma_{ij}\) and \(\tau_{ij}\) are the interaction and turbulent momentum flow tensors, and

\[
\begin{align*}
\varsigma_{ij} &= \rho^{-1} \varepsilon_{ilk} \left[ <x_l u_k> - <x_l><u_k> - <x_l><u_k> u_j> - <x_l>< u_j> u_k> \right] \quad \text{(13)} \\
\beta_i &= \varepsilon_{ilk} \left[ <x_l b_k> - <x_l>< b_k> \right] \quad \text{(15)}
\end{align*}
\]

are respectively the inertial (containing stretching) and interaction flow tensors of angular momentum and \(\beta\) is the couple associated to the external field \(b\).

In this system the terms that need to be represented through a model are the turbulent momentum and angular momentum stresses. The functional relations on which the model relies are all of them Galilean invariants and are listed below:

\[
\begin{align*}
\nu_h &= ch, \\ 
\tau_{ij} &= ch (\partial_j <u_i> + \partial_i <u_j> - \frac{2}{3} \partial_k <u_k> \delta_{ij}) \\ 
c_{ij} &= <u_i> h_j + ch (\partial_j h_i + \partial_i h_j - \frac{2}{3} \partial_k h_k \delta_{ij})
\end{align*}
\]

The first term on the right hand side of (18) represents the role played by the stretching, while the other simulates the momentum transfer due to the turbulent convection. The todays reference large eddy simulation method is based on the adoption of Smagorinsky’s\(^\text{18}\) or the vorticity\(^\text{19}\) models, which assume a local invariance of the turbulent motion. Thus in the immediate nearness of some point in time and space a dynamical similarity is
assumed everywhere in the field. The field nondimensionalization is based on the existence of local turbulent scales small enough to adjust to the slowly changing environment in the external scale. With this model we introduce one degree of freedom - the intrinsic angular momentum - that is portrayed by a relevant differential equation, coupled however independent from the momentum equation. In this way we hope to be able to simulate also turbulent flow not in local equilibrium. Of course this would depend from the propriety with which the turbulent flow tensor of the intrinsic moment of momentum is modeled. In relation (18) we tried to insert the two major inertial phenomena present at the level of the subgrid scales, the stretching and the transport due to the turbulent convection. Notwithstanding the introduction of an additional differential equation, in the model one subgrid constant $c$ only appears. Assuming that the largest resolvable wave number lies within the inertial range, that the energy transferred from the resolved scales to the subgrid scales is equal to the energy dissipated by the these last and that the energy of the subgrid scales is that owned by their inertial part, see Lilly, Yoshizawa, constant $c$ may be estimated as 0.066, see the Appendix for details. This value has been numerically verified by carrying a priori tests over realizations of homogeneous isotropic velocity fields. Note that in local turbulence equilibrium conditions the scaling of the turbulent viscosity, with respect to the dissipation function $\varepsilon$ and the filtering length $\delta$, is the same as that of the intrinsic angular momentum

$$h \sim \delta^{4/3} \varepsilon^{1/3} \sim \nu_{\delta}. \quad (19)$$

For the derivation of these scaling laws see Monin and Yaglom in regard to $h$ and Yoshizawa, Leslie and Quarini in regard to $\nu_{\delta}$. In synthesis the main lineaments of this model are: the capability to follow the evolution of $h$, and thus of $\nu_{\delta}$, through a relevant differential equation and the proper scaling with respect to both the filtering length and the dissipation rate. The differential nature suggest an employment for simulations of non equilibrium turbulent flows.

A unique feature of the present model is its natural convenience to the simulation of structured fluid in turbulent motion. In this case being the coupling between the momentum and angular momentum equations already introduced by the physics of the system the model reduces from differential to algebraic.

The detailed validation of the model, based on the energy decay and spectra properties of isotropic homogeneous decaying turbulence, has been carried out in Iovieno. In here, simulations of turbulence shearless mixings with strong integral scale non-homogeneity were successfully contrasted with direct numerical and experimental simulations. Simulations of wall turbulence will be posted in a future work.

4. Conclusions

By means of a series expansion in terms of $\delta^2$ - $\delta$ being the linear dimension of average - we determined a new representation of the averaged angular momentum balance in terms of an infinite sequence of indipendent differential equations where linear antisymmetric operators act on the momentum. The first term of the sequence is just the Helmholtz equation. The others may be viewed as a kind of high order vorticity equations and could be used in turbulence mechanics as auxiliar equations to describe the evolution of
the correlation variables coming out from the filtering of the turbulent equations. This representation may be also applied to the motion of structured fluid as long as they may be considered as locally homogeneous.

The spatial filtering is not able to introduce asymmetries in homogeneous flows, as well in turbulent motion. The application to turbulent flows of models suited to flows of a structured fluid and characterized by the coupling of the momentum and angular momentum equations, through the presence of an antisymmetric part of the stress tensor, is not in our opinion justified.

We propose a new differential large eddy scale model based on a different kind of coupling of the momentum and angular momentum equations, that relies on the assumption of a turbulent diffusivity proportional to the intrinsic moment of momentum. This coupling do not spoil the symmetry property of the stress tensor. The model shows a proper scaling of the eddy diffusivity with respect either to the filtering length or the dissipation function and thus to the integral scale of the motion and it contains one subgrid scale coefficient only. In case of turbulent motion of a structured fluid, where the coupling is already present owing the physical nature of the problem, the model becomes algebraic.

Appendix: Evaluation of the model constant

In this section the costant $c$ of the model, defined by equation (16) of Sec. 3.2, is estimated assuming that the largest resolvable wave number $2\pi/\delta$ lies within the inertial range, that the energy transfer rate from the resolved scales to the subgrid scales is equal to the energy dissipation rate $\varepsilon$ and that a great separation of scales exists. In such a situation the energy of the subgrid scales is mostly that owned by their inertial part. Under this assumption Yoshizawa$^{21}$ determined the constant of the scaling law for the eddy-viscosity

$$\nu_\delta = c_\nu \varepsilon^{4/3} \delta^{1/3}, \quad c_\nu \approx 0.053.$$ 

Considering spherical volumes, of radius $\delta$, of averages we may write the intrinsic angular momentum $h$, introduced in Sec. 2, as

$$h \approx \frac{3}{4\pi \delta^3} \int_0^\delta u_\lambda 4\pi \lambda^3 d\lambda,$$

(20)

where $u_\lambda$ is the turbulent velocity variation over distances of the order $\lambda$. The Kolmogorov’s law yields

$$u_\lambda = (3\alpha)^{\frac{1}{2}} \left( \frac{\varepsilon \lambda}{2\pi} \right)^{\frac{1}{3}},$$

where $\alpha$ is the Kolmogorov’s constant, approximately equal$^{20}$ to 1.5. Integral (20) leads then to

$$h = c_h \varepsilon^{4/3} \delta^{1/3}, \quad c_h = \frac{9(3\alpha)^{\frac{1}{2}}}{13(2\pi)^{\frac{1}{3}}} \approx 0.80$$

and the constant $c$ in (16) is consequently

$$c = \frac{c_\nu}{c_h} \approx 0.066,$$

a value which has been successfully confirmed through a priori numerical test on homogeneous isotropic turbulence$^{22,23}$. 

8
References


