EFMC-2006: The intermediate asymptotics of turbulent diffusion

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Shearless turbulence mixing: an overview

- mixing between homogeneous turbulences: no mean shear \(\Rightarrow\) no turbulence production

- the mixing layer is generated by the turbulence inhomogeneity, i.e.:
  - by the gradient of turbulent energy
  - by the gradient of integral scale
Properties of laboratory/numerical numerical experiments:

- A self-similar stage of decay is always reached
- It is characterized by a strong intermittent penetration, which depends on the two mixing parameters:
  - the turbulent energy gradient
  - the integral scale gradient

This behaviour must be contained in the solutions of:

- the two-point correlation equation which allows to consider both the macroscale and energy gradient parameters
  \[ \langle B_{ij}(x, r, t) \rangle = u_i(x, t)u_j(x + r, t) \];
- the one-point correlation equation, the limit \( r \to 0 \), which allows to obtain the third order moment (skewness) distribution from second order (energy) distribution.
Intermittency: skewness and kurtosis profiles

\[ S = \frac{\overline{u^3}}{\overline{u^2}^3}, \quad K = \frac{\overline{u^4}}{\overline{u^2}^2} \quad \Rightarrow \quad S \approx 0, \quad K \approx 3 \text{ in homogeneous isotropic turb.} \]

\[ \mathcal{E} = 6.6, \quad \mathcal{L} = 1.5 : \text{gradients of energy and scale have the same direction.} \]
Intermittent penetration $\eta_{max}$ - maximum of skewness $\max\{S\}$

for $L = 1$ we have the approximate scaling law

$$\eta_{max} \sim a(\mathcal{E} - 1)^b, \quad a = 0.27, \quad b = 0.33,$$

$$\max\{S\} \sim a(\mathcal{E} - 1)^b, \quad a = 0.46, \quad b = 0.31$$

main reference: Tordella and Iovieno *JFM* 549 (2006)
Two-point double correlations:

\[
B_{ij}(x, r, t) = u_i(x, t)u_j(x + r, t)
\]
\[
B_{pi}(x, r, t) = p(x, t)u_i(x + r, t)
\]
\[
B_{ip}(x, r, t) = u_i(x, t)p(x + r, t)
\]

Two-point triple correlations:

\[
B_{ij|k}(x, r, t) = u_i(x, t)u_j(x, t)u_k(x + r, t)
\]
\[
B_{i|jk}(x, r, t) = u_i(x, t)u_j(x + r, t)u_k(x + r, t)
\]

Momentum equations:

\[
\frac{\partial}{\partial t}B_{ij}(x, r, t) + \frac{\partial}{\partial x_k}B_{ik|j}(x, r, t) + \frac{\partial}{\partial r_k}\left(B_{i|kj}(x, r, t) - B_{i|kj}(x, r, t)\right) =
\]
\[
= \frac{1}{\rho}\left\{-\frac{\partial}{\partial x_i}B_{pj}(x, r, t) + \frac{\partial}{\partial r_i}B_{pj}(x, r, t) - \frac{\partial}{\partial r_j}B_{ip}(x, r, t)\right\} +
\]
\[+ \nu \left[ \frac{\partial^2}{\partial x_k \partial x_k} \frac{\partial}{\partial t}B_{ij}(x, r, t) + 2 \frac{\partial^2}{\partial r_k \partial r_k} \frac{\partial}{\partial t}B_{ij}(x, r, t) - 2 \frac{\partial^2}{\partial x_k \partial r_k} \frac{\partial}{\partial t}B_{ij}(x, r, t) \right] \]
Simmetries:

- there is only one non homogeneous and non isotropic direction, denoted by $x$
- correlations are invariant under all translations perpendicular to $x$ and to all rotations around $x$

$\Rightarrow$ with cylindrical coordinates all variables are functions of $(x, r_0, r_x, t)$ only

$B_{\partial_x u_x \partial_x u_x}(x, r_0, 0, t) = \lambda^{-2}(x, r_0, t)B_{xx}(x, r_0, 0, t)$

Two-point lateral correlation $B_{xx}$ equation for $r_x \rightarrow 0$ reduces to

$$\frac{\partial}{\partial t}B_{xx} + \frac{\partial}{\partial x}B_{xx}|_x - 2 \left( \frac{\partial B_{rx}|_x}{\partial r_0} + \frac{B_{rx}|_x}{r_0} \right) =$$

$$= -\frac{1}{\rho \partial x}B_{px} + \nu \left\{ \left[ \frac{\partial^2}{\partial x^2} + 2 \left( \frac{\partial^2}{\partial r_0^2} + \frac{1}{r_0 \partial r_0} \right) \right] B_{xx} - \frac{B_{xx}}{\lambda^2(x, r_0, t)} \right\} (1)$$
Hypothesis and simplifications

• The two homogeneous turbulences decay in the same way, thus

\[ E_1(t) = A_1(t + t_0)^{-n_1}, \quad E_2(t) = A_2(t + t_0)^{-n_2} \]

the exponents \( n_1, n_2 \) are close each other (numerical experiments, Tordella & Iovieno, *JFM* 2006). Here, we suppose \( n_1 = n_2 = n = 1 \), a value which corresponds to \( R_\lambda \gg 1 \) (Batchelor & Townsend, 1948).

⇒ this implies \( \mathcal{E} = E_1(t)/E_2(t) = \text{const}, \quad \mathcal{L} = \ell_1(t)/\ell_2(t) = \text{const} \)

• Pressure-velocity correlation has been shown to be approximately proportional to the convective fluctuation transport (Yoshizawa, 1982, 2002) for \( \mathbf{r} \to 0 \), so that

\[ -\overline{pu_i} = a \rho \overline{u_i u_i u_j} \frac{1}{2}, \quad a \approx 0.10 \]

so that

\[ -B_{px}(x, 0, 0, t) = 0.25 \rho B_{xx|x}(x, 0, 0, t) \]
Similarity hypothesis

The two-point lateral correlations are determined by

- the coordinates $x, r_0, t$
- the energies $E_1(t), E_2(t) \Rightarrow \text{gradient } \nabla E$ or their ratio $\mathcal{E} = E_1/E_2$
- the integral scales $\ell_1(t), \ell_2(t) \Rightarrow \text{gradient } \nabla \ell$ or their ratio $\mathcal{L} = \ell_1/\ell_2$
- the mixing layer thickness $\Delta(t)$

The turbulent kinetic energy and the integral scale of the high energy region $(E_1, \ell_1)$, that is

$$B_{xx}(-\infty, 0, 0, t) = \frac{2}{3} E_1(t), \quad \ell_1(t)$$

are chosen as velocity and length scales.
Consequently all correlations can be expressed in terms of

\[ \eta = \frac{x}{\Delta(t)}, \quad \xi = \frac{r_0}{\ell(x, t)}, \]

and this is the structure of the equations for the similarity analysis:

\[
\begin{align*}
B_{xx}(x, r_0, 0, t) &= B_{xx}(-\infty, 0, 0, t) \varphi_{xx}(\eta, \xi) \\
B_{xx|x}(x, r_0, 0, t) &= B_{xx}^{\frac{3}{2}}(-\infty, 0, 0, t) \varphi_{xx|x}(\eta, \xi) \\
B_{rx|x}(x, r_0, 0, t) &= B_{xx}^{\frac{3}{2}}(-\infty, 0, 0, t) \varphi_{rx|x}(\eta, \xi) \\
B_{px}(x, r_0, 0, t) &= \rho B_{xx}^{\frac{3}{2}}(-\infty, 0, 0, t) \varphi_{px}(\eta, \xi) \\
\lambda(x, r_0, t) &= \ell_1(t) \tilde{\lambda}(\eta, \xi) \\
\ell(x, t) &= \ell_1(t) \tilde{\ell}(\eta)
\end{align*}
\]
⇒ similarity conditions:

By introducing the similarity relations in the equation for $B_{xx}$ and by imposing that all the coefficients must be independent from $(x, r_0, t)$, the following condition is obtained

$$\Delta(t) \propto \ell_1(t) = \mathcal{L} \ell_2(t)$$

Equation (1) reduces to

$$-\varphi_{xx} + \frac{1}{2} \left[ \psi \frac{\partial \varphi_{xx}}{\partial \eta} - \xi \varepsilon \frac{\partial \varphi_{xx}}{\partial \xi} \right] + \frac{3}{4f(R_{\ell_1})} \left[ \frac{\partial \varphi_{xx|\eta}}{\partial \eta} - \xi \varepsilon \frac{\partial \varphi_{xx|\eta}}{\partial \xi} - 2 \frac{1}{\ell_2 \xi \xi} \left( \xi \varphi_{xx|\eta} \right) \right] =$$

$$= -\frac{3}{f(R_{\ell_1})} \left[ \frac{\partial \varphi_{xx}}{\partial \eta} - \xi \varepsilon \frac{\partial \varphi_{xx}}{\partial \xi} \right] + \frac{3}{2f(R_{\ell_1}) \ell_1} \left\{ \left[ \frac{\partial^2 \varphi_{xx}}{\partial \eta^2} - \xi \varepsilon \left( \frac{\partial^2 \varphi_{xx}}{\partial \eta \partial \xi} - \varphi_{xx} \xi \varepsilon \right) \right] - \xi \varepsilon \frac{\partial \varphi_{xx}}{\partial \xi} \right\}$$

boundary conditions: matching with the two homogeneous turbulences external to the mixing:

- for $\eta \to -\infty$ to homogeneous turbulence with energy $E_1$ and scale $\ell_1$
- for $\eta \to +\infty$ to homogeneous turbulence with energy $E_2$ and scale $\ell_2$
One-point limit

For $\xi \rightarrow 0$, this equation and its boundary conditions become

$$
\frac{\partial \varphi_{xx}|_x}{\partial \eta} = 4 f(R_{\ell_1}) \left\{ \frac{1}{2} \frac{\partial \varphi_{xx}}{\partial \eta} + \frac{3}{2} \frac{1}{f(R_{\ell_1}) R_{\ell_1}} \frac{\partial^2 \varphi_{xx}}{\partial \eta^2} \right. \\
+ \left. \varphi_{xx} \left[ 1 - \frac{3}{f(R_{\ell}(\eta)) R_{\ell}(\eta)} \frac{\tilde{\ell}^2(\eta)}{\bar{\lambda}^2(\eta, 0)} \right] \right\}
$$

where $R_{\ell}(\eta)$ is the local Reynolds number, i.e. based on local energy and scale

**second order correlation b.c.:**

$$
\lim_{\eta \rightarrow -\infty} \varphi_{xx}(\eta, 0) = \frac{2}{3}, \quad \lim_{\eta \rightarrow +\infty} \varphi_{xx}(\eta, 0) = \frac{2}{3} \mathcal{E}^{-1},
$$

**third order correlation b.c.:**

$$
\lim_{\eta \rightarrow \pm\infty} \varphi_{xx}|_x(\eta, 0) = 0
$$

This equation relates the distributions of second and third order moments in the mixing layer.
The flow outside the mixing (|\eta| \gg 1) is homogeneous, thus
\[ \frac{\partial}{\partial \eta} = 0 \Rightarrow \left[ 1 - \frac{3}{f(R_\ell(\eta))} \frac{1}{R_\ell(\eta)} \tilde{\ell}^2(\eta) \right] = 0 \quad \text{for} \quad \eta \to \pm \infty \quad \text{(♣)} \]
this is consistent with homogeneous turbulence, where it is known that
\[ \left( \frac{\ell}{\chi} \right)^2 \propto R_\ell. \]

However, this is not necessarily true through the mixing layer:

- when \( \mathcal{L} = 1 \) there is still equilibrium as regards the integral scale, so we assume that this relation still holds
- when \( \mathcal{L} \neq 1 \) we suppose that function (♣) is a function of the scale gradients, so that we could write
\[ \tilde{\lambda}(\eta, 0) = \left( \frac{3}{f(R_\ell(\eta))R_\ell(\eta)} \right)^{\frac{1}{2}} \tilde{\ell}(\eta) \left( 1 - \frac{b}{\varphi_{xx} \, d^2\ell/\, d\eta^2(\eta)} \right)^{-\frac{1}{2}} \]
Now, the distribution of skewness can be obtained from the one-point correlation equation when a typical expression for the second order moment distribution is introduced as in Veeravalli & Warhaft *JFM* 1989:

\begin{align*}
\varphi_{xx}(\eta) &= 1 + \mathcal{E}^{-1} - \frac{1 - \mathcal{E}^{-1}}{2} \text{erf} (\eta) \\
\tilde{\ell}(\eta) &= 1 + \mathcal{L}^{-1} - \frac{1 - \mathcal{L}^{-1}}{2} \text{erf} (a\eta)
\end{align*}

The third order moment is then

\[
\varphi_{xxx} = \frac{f}{6} \frac{1 - \mathcal{E}^{-1}}{\sqrt{\pi}} \left[ \left( 1 - \frac{6}{f(R\ell_1)} \right) e^{-\eta^2} + ba(1 - \mathcal{L}^{-1})e^{-(a\eta)^2} \right].
\]

Parameters $a > 0$ and $b > 0$ are chosen to fit the experimental behaviour
Normalized energy and skewness distributions: $\mathcal{E} = 6.7$ and $\mathcal{L} = 1$. 

(a) 

(b) 

$\tau_1$ 

3.7 

4.5 

5.2 

5.6 

$\pm 0.0$ 

0.2 

0.4 

0.6 

0.8 

1.0 

$x-x_m/\Delta$ 

0 

1 

2 

3 

$\nu/\tau_1$ 

$\phi_{uu}$ 

Similarity profile

$\pm 0.2$ 

0.0 

0.2 

0.4 

0.6 

0.8 

1.0 

$x-x_c/\Delta$ 

0 

1 

2 

3 

$S$ 

LES – IAM model 

DNS, Briggs et al. 

Similarity solution
Position of the skewness maximum

\[ \eta_{\text{max}} \]

Scale ratio

\[ \frac{E_1}{E_2} \]

exp. fitting \( L=1 \)

\[ \text{Scale ratio} \]

- \( 1.7 \)
- \( 1.4 \)
- \( 1.2 \)
- \( 1.0 \)
- \( 0.8 \)
- \( 0.6 \)

Energy ratio \( \frac{E_1}{E_2} \)

Graph showing the position of the skewness maximum with varying scale ratios.
Conclusions

The intermediate asymptotics of the turbulence diffusion in the absence of production of turbulent kinetic energy is considered.

• An intermediate similarity stage of decay always exists.
• When the energy ratio $E$ is far from unity, the mixing is very intermittent.
• When $L = 1$, the intermittency increases with the energy ratio $E$ with a scaling exponent that is almost equal to 0.3.
• Intermittency smoothly varies when passing through $L = 1$:
  it increases when $L > 1$ (*concordant* gradient of energy and scale),
  it reduces when $L < 1$ (*opposite* gradient of energy and scale)
• The similarity solution contains all the salient points showed by the experiments.

→ Animations (*kinetic energy*): $L = 0.6$  $L = 2.1$
Similarity equation

\[-\varphi_{xx} + \frac{1}{2} \left[ \eta \frac{\partial \varphi_{xx}}{\partial \eta} - \xi \eta \tilde{\ell}' \frac{\partial \varphi_{xx}}{\partial \xi} \right] + \]

\[= \frac{3}{4 f(R_{\ell_1})} \left[ \frac{\partial \varphi_{xx}}{\partial \eta} \right] - \xi \tilde{\ell}' \frac{\partial \varphi_{xx}}{\partial \xi} - 2 \frac{1}{\tilde{\ell} \xi} \frac{\partial}{\partial \xi} \left( \xi \varphi_{rx|x} \right) \]

\[= -\frac{3}{f(R_{\ell_1})} \left[ \frac{\partial \varphi_{px}}{\partial \eta} - \xi \tilde{\ell}' \frac{\partial \varphi_{px}}{\partial \xi} \right] + \]

\[+ \frac{3}{2 f(R_{\ell_1}) R_{\ell_1}} \left\{ \left[ \frac{\partial^2 \varphi_{xx}}{\partial \eta^2} - \xi \tilde{\ell}' \left( \frac{\partial^2 \varphi_{xx}}{\partial \eta \partial \xi} - \tilde{\ell}' \frac{\partial \varphi_{xx}}{\partial \xi} - \xi \tilde{\ell}' \frac{\partial^2 \varphi_{xx}}{\partial \xi^2} \right) \right] - \xi \tilde{\ell}'' \frac{\partial \varphi_{xx}}{\partial \xi} \right\} \]

\[+ \left[ \frac{2}{\tilde{\ell}^2 \xi} \frac{\partial}{\partial \xi} \left( \xi \frac{\partial \varphi_{xx}}{\partial \xi} \right) - \frac{\varphi_{xx}}{\tilde{\chi}^2} \right] \} \]