Dispersive equations and their role beyond PDE

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Introduction

In this talk I will use the periodic semi linear Schrődinger Cuachy problem

$$\begin{cases} iu_t + \Delta u = \lambda |u|^{p-1} u, \\ u(x,0) = u_0(x), \quad x \in \mathbb{T}^n, \end{cases} \tag{1.1}$$

where $u_0(x)$ is the initial profile, $p > 1$, $u : \mathbb{R} \times \mathbb{T}^n \to \mathbb{C}$, and $\mathbb{T}^n$ is a n-dimensional torus[1], to illustrate how a partial differential equation that may have been introduced to model a certain phenomenon in physics may also have structures that touch many different areas of mathematics like Fourier and harmonic analysis, analytic number theory, probability, dynamical systems and symplectic geometry.

For obvious reasons here I will only present the *simple* connections of problem (1.1) with the areas of mathematics listed above and one should not think that this is all there is. In the contrary all of these connections are very active areas of research at the moment with many open problems.

[1] Later we will distinguish between a rational and an irrational torus.

Bose Einstein Condensate

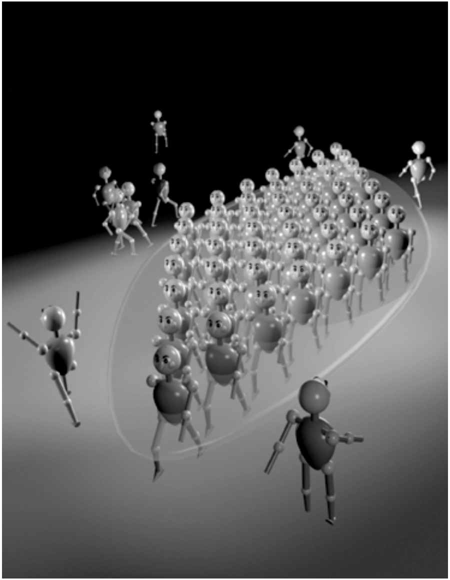
The Cauchy problem

$$\begin{cases} iu_t + \Delta u = \lambda |u|^2 u, \\ u(x,0) = u_0(x), \quad x \in \mathbb{T}^3 \end{cases} \tag{1.2}$$

is used to describe several phenomena, but in particular is the problem that governs the Bose Einstein Condensate (BEC). The BEC is the state of matter of a dilute gas of weakly interacting bosons confined in an external potential and cooled to temperatures very near absolute zero. The point wise density of this *gas* at time t is represented by $u(x,t)$ in the problem above.

Physically what happens is that the bosons particles, that are in a chaotic state when the temperature T is above absolute zero, as $T \to 0$ they start rearranging themselves, losing their individual identity and shaping themselves as a wave $u(x,t)$ that solves an equation as the one in (1.2).

Rendering of Bose Einstein Condensate



Science (December 22, 1995) by Steve Keller.

The Basic Questions

When a Cauchy problem is proposed as a model the obvious question to study is *well-posedness*, that is

- Existence of solutions
- Uniqueness of solutions
- Stability of solutions

One way to address the question of well-posedness is by rewriting the Cauchy problem (1.1) as the integral equation given by the Duhamel principle

$$u(x,t) = S(t)u_0(x) + c\int_0^t S(t-t')|u|^{p-1}u(x,t'))\,dt', \tag{1.3}$$

where $S(t)u_0(x)$ is the solution of the associated linear problem that we will discuss in details below. The idea then is to use a fixed point theorem in a space of functions which norm is dictated by strong estimates for $S(t)u_0(x)$.

The linear solution $S(t)u_0(x)$ in $\mathbb{T}$

We first recall that $S(t)u_0(x) = v(x,t)$, where $v(x,t)$ solves the Cauchy problem

$$\text{(1.4)} \qquad \begin{cases} iv_t + \Delta v = 0, \\ v(x,0) = u_0(x), \end{cases}$$

for simplicity we assume that $x \in \mathbb{T}$. Using Fourier series we write the solution

$$v(x,t) = \sum_{k \in \mathbb{Z}} a(k,t)e^{ikx}, \quad \text{and} \quad a(k,t) = \hat{v}(k,t).$$

If we take the Fourier transform of (1.4), for every frequency k we obtain an ODE for $a(k,t)$:

$$\text{(1.5)} \qquad \begin{cases} i\frac{d}{dt}a(k,t) + (-ik)^2 a(k,t) = 0, \\ a(k,0) = \hat{u}_0(k), \end{cases}$$

and the solution becomes

$$a(k,t) = \hat{u}_0(k)e^{itk^2} \quad \text{and} \quad v(x,t) = \sum_{k \in \mathbb{Z}} a(k,t)e^{i(kx+tk^2)}.$$

The linear solution $S(t)u_0$ in $\mathbb{T}^n$

Now we assume that $x \in \mathbb{T}^n$ and that $c_i > 0,\ i = 1, ..., n$ are the periods. Then if we repeat the same argument we obtain that

$$S(t)u_0(x,t) = \sum_{k \in \mathbb{Z}} a(k,t) e^{i(kx + t\gamma(k))},$$

where

$$\gamma(k) = \sum_{i=1}^{n} c_i k_i^2.$$

It will be relevant for later to observe that in the special case when $c_i = 1,\ i = 1, ..., n$

$$\gamma(k) = R^2$$

represents the sphere of radius R.
If $c_i \in \mathbb{N},\ i = 1, ..., n$ we will call the torus $\mathbb{T}^n$ a *rational* torus, otherwise we will call it *irrational*.

Strichartz Estimates

The Strichartz estimates in $\mathbb{T}^n$ are non trivial estimates for $S(t)u_0(x)$. They where originally introduced by Bourgain as a conjecture that then he partially resolved.

Conjecture

Assume that $\mathbb{T}^n$ is a rational torus and the support of $\hat{u}_{0,N}$ is in the annulus $\{|n| \lesssim N\}$. Then

$$\|S(t)u_{0,N}\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \lesssim C_q \|u_{0,N}\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q < \frac{2(n+2)}{n}$$

$$\|S(t)u_{0,N}\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \ll N^{\epsilon} \|u_{0,N}\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q = \frac{2(n+2)}{n}$$

$$\|S(t)u_{0,N}\|_{L^q_t L^q_x([0,1]\times\mathbb{T}^n)} \lesssim C_q N^{\frac{n}{2}-\frac{n+2}{q}} \|u_{0,N}\|_{L^2_x(\mathbb{T}^n)} \quad \text{if} \quad q < \frac{2(n+2)}{n}$$

The L^4 Strichartz Estimates in $\mathbb{T}^2$

Let's concentrate on the L^4 Strichartz estimate that Bourgain proved in the 90's:

$$\|S(t)u_{0,N}\|_{L^4_{\mathbb{T}}L^4_T} \leq N^{\epsilon}\|u_{0,N}\|_{L^2_x} \quad \Longrightarrow \quad \|S(t)u_0\|_{L^4_{\mathbb{T}}L^4_T} \leq \|u_0\|_{H^{\epsilon}_x}.$$

We are not going to repeat the proof, here we only say that it is based on counting $\mathbb{Z}^2$ lattice points on ellipses given by

$$\gamma(k) = c_1 k_1^2 + c_2 k_2^2 = R^2$$

for $R \gg 1$. It is here that the rationality of the torus comes into play. In fact if the torus is rational, that is $c_j \in \mathbb{N}$, one can use some standard results from analytic number theory and obtain the sharp bound

$$\#\{k \in \mathbb{Z}^2 / \gamma(k) = R^2\} \sim expC \frac{\log R}{\log \log R}.$$

For irrational tori, that is when we simply assume $c_j \in \mathbb{R}^+$, only partial results are known so far, see **Bourgain, Burq-Gerard-Tzvetkov, Hani**.

Local and global solutions

The norm L^2 that appear on the right hand side of the Strichartz estimates is not there by chance. In fact for the Schrödingier equation

$$iu_t + \Delta u = \lambda |u|^{p-1} u$$

the integral

$$M(u(t)) = \int |u(x,t)|^2 \, dx$$

is in fact the *Mass* and it is conserved. As a consequence the most natural space for the initial data u_0 is the L^2 space. Unfortunately though at this level of regularity often little can be done to control nonlinear interactions and obtain well-posedness.
The next most relevant space is the Sobolev H^1 space. In fact the equation above keeps also the Energy (Hamiltonian) conserved:

$$H(u(t)) = \frac{1}{2} \int |\nabla u|^2(x,t) \, dx - \frac{2\lambda}{p+1} \int |u(t,x)|^{p+1} \, dx$$

Focusing and Defocusing

$$H(u(t)) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx - \frac{2\lambda}{p+1}\int |u(t,x)|^{p+1}\,dx$$

- If $\lambda = 1$ (Focusing) the energy could be negative and blow up may occur.
- If $\lambda = -1$ (Defocusing) the energy and the mass can be used to obtain a global in time a priori bound for the the H^1 norm of the solution $u(x,t)$. One could use this a priori bound and Strichartz estimates to get theorems as this one:

Theorem (Bourgain)

The Cauchy problem

$$\begin{cases} iu_t + \Delta u = -|u|^2 u, \\ u(x,0) = u_0(x), \quad x \in \mathbb{T}^2 \quad \textit{rational} \end{cases}$$

is globally well-posed for data u_0 in H^1.

Now that we know that this equation has a global flow $u_0 \to u(x,t)$ we can start asking questions on the behavior of $u(x,t)$ in time.

Notion of Weak Turbulence

Definition

For this talk *Weak Turbulence* is the phenomenon of global-in-time solutions shifting their support towards increasingly high frequencies.

This shift is also called forward cascade.

- One way of measuring weak turbulence is to consider the function in time

$$\|u(t)\|_{\dot{H}^s}^2 = \int |\hat{u}(t,k)|^2 |k|^{2s} dk$$

  for $s \gg 1$ and prove that it grows for large times t.
- Weak turbulence is incompatible with scattering or complete integrability.

There are two theorems that summarize the state of the affairs in this context. The first gives some polynomial in time bounds for $\|u(t)\|_{\dot{H}^s}^2$, the second[2] shows some kind of growth for certain solutions to the Cuachy problem above.

[2] Although for a different equation, there are some recent interesting results in this context by **Pocovnicu**.

Two theorems on weak turbulence

Theorem (**Bourgain**, **Sohinger**)

Let u be the global solution of the cubic, defocusing, NLS equation on a rational $\mathbb{T}^2$:

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^2 u \\ u(0,x) = u_0(x), \quad where\ x \in \mathbb{T}^2, \end{cases} \tag{3.1}$$

Then

$$\|u(t)\|_{H^s(\mathbb{T}^2)} \lesssim (1+|t|)^{s+} \|u_0\|_{H^s(\mathbb{T}^2)}.$$

Theorem (**Colliander-Keel-Staffilani-Takaoka-Tao**)

Let $s > 1$, $K \gg 1$ and $0 < \sigma < 1$ be given. Then there exist a global smooth solution $u(x,t)$ to the defocusing IVP (3.1) above and $T > 0$ such that $\|u_0\|_{H^s} \leq \sigma$ and $\|u(T)\|_{H^s}^2 \geq K$.

The toy model

Here we only give few ideas that are relevant for the proof of the second theorem.
We start by making the ansatz

$$v(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(\langle n,x\rangle + |n|^2 t)},$$

and by rewritint the equation as an ODE in terms of the infinite vector $(a_n(t))$. We also consider only the resonant part of the ODE and we construct a special finite set of frequencies Λ that is closed under resonant interactions and has several other “good” properties. Thanks to these properties we arrive to a finite dimension toy model

$$-i\partial_t b_j(t) = -b_j(t)|b_j(t)|^2 - 2b_{j-1}(t)^2\overline{b_j(t)} - 2b_{j+1}(t)^2\overline{b_j(t)},$$

for $j = 0, ..., M+1$, with the boundary condition

$$b_0(t) = b_{M+1}(t) = 0.$$

Remark

This new IVP conserves the momentum, the mass ($\sum_{j=1}^{M} |b_j(t)|^2 = 1$) and the energy!

Global well-posedness for this system is not an issue. Then we define

$$\Sigma = \{x \in \mathbb{C}^M \ / \ |x|^2 = 1\} \text{ and } \tilde{W}(t) : \Sigma \to \Sigma,$$

where $\tilde{W}(t)b(0) = b(t)$ for any solution $b(t)$ of our system. It is easy to see that if we define the torus

$$\mathbb{T}_j = \{(b_1, ..., b_M) \in \Sigma \, / \, |b_j| = 1, \; b_k = 0, \; k \neq j\}$$

then

$$\tilde{W}(t)\mathbb{T}_j = \mathbb{T}_j \text{ for all } j = 1, ..., M$$

($\mathbb{T}_j$ is invariant).

At this point the problem has been set up in such a way that if we could show that once we start "near" one of the first tori (low frequencies) we end up at a certain time T near one of the last tori (high frequencies) then we are done. In fact we have the following result:

Theorem (Sliding theorem)

Let $M \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of $\mathbb{T}_3$ and x_{M-2} within ϵ of $\mathbb{T}_{M-2}$ and a time T such that

$$W(T)x_3 = x_{M-2}.$$

What the theorem says is that $W(t)x_3$ is a solution of total mass 1 arbitrarily concentrated near mode $j = 3$ at some time 0 that gets moved so that it is concentrated near mode $j = N - 2$ at later time T.

The complete proof is lengthy. Here we only give a motivation for it that should clarify the dynamics involved.
Let us first observe that when $M = 2$ we can easily demonstrate that there is an orbit connecting $\mathbb{T}_1$ to $\mathbb{T}_2$. Indeed in this case we have the explicit "slider" solution

$$b_1(t) := \frac{e^{-it}\omega}{\sqrt{1 + e^{2\sqrt{3}t}}}; \quad b_2(t) := \frac{e^{-it}\omega^2}{\sqrt{1 + e^{-2\sqrt{3}t}}}$$

where $\omega := e^{2\pi i/3}$ is a cube root of unity.
This solution approaches $\mathbb{T}_1$ exponentially fast as $t \to -\infty$, and approaches $\mathbb{T}_2$ exponentially fast as $t \to +\infty$.

One can translate this solution in the j parameter, and obtain solutions that "slide" from $\mathbb{T}_j$ to $\mathbb{T}_{j+1}$. Intuitively, the proof of the Sliding Theorem for higher M should then proceed by concatenating these slider solutions....This though cannot work directly because each solution requires an infinite amount of time to connect one hoop to the next. It turned out though that a suitably perturbed or "fuzzy" version of these slider solutions can in fact be glued together.

Finite Dimension Hamiltonian Systems

Hamilton's equations of motion have the antisymmetric form

$$\dot{q}_i = \frac{\partial H(p,q)}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H(p,q)}{\partial q_i}$$

the Hamiltonian $H(p,q)$ being a first integral:

$$\frac{dH}{dt} := \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = \sum_i \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = 0.$$

By defining $y := (q_1, \ldots, q_k, p_1, \ldots, p_k)^T \in \mathbb{R}^{2k}$ ($2k = d$) we can rewrite the system in the compact form

$$\frac{dy}{dt} = J\nabla H(y), \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

We have

Theorem (Liouville's Theorem)

Let a vector field $f : \mathbb{R}^d \to \mathbb{R}^d$ *be* **divergence free**. *If the flow map* Φ_t *satisfies*

$$\frac{d}{dt}\Phi_t(y) \;=\; f(\Phi_t(y)),$$

then Φ_t *is a volume preserving map for all* t.

In particular if f is associated to a Hamiltonian system then automatically $div\, f = 0$. As a consequence the Lebesgue measure ν on $\mathbb{R}^{2k}$ is invariant under the Hamiltonian flow:

$$\nu(\Phi_t(A)) = \nu(A)$$

for all measurable sets A.

Finite Dimension Gibbs Measure

A more interesting measure is the Gibbs measure. We have in fact:

Theorem (Invariance of Gibbs measures)

Assume that Φ_t is the flow generated by the Hamiltonian system above. Then the Gibbs measures defined as

$$d\mu := e^{-\beta H(p,q)} \prod_{i=1}^{d} dp_i \, dq_i$$

with $\beta > 0$, are invariant under the flow Φ_t.

The proof is trivial since from conservation of the Hamiltonian H the functions $e^{-\beta H(p,q)}$ remain constant, while, thanks to Liouville's Theorem the volume $\prod_{i=1}^{d} dp_i \, dq_i$ remains invariant as well.

Infinite Dimension Hamiltonian systems

Consider the Cauchy problem

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^4 u \\ u(0,x) = u_0(x), \text{ where } x \in \mathbb{T}. \end{cases}$$

with Hamiltonian

$$H(u(t)) = \frac{1}{2}\int |\nabla u|^2(x,t)\,dx + \frac{1}{3}\int |u(t,x)|^6\,dx.$$

One can rewrite the Cauchy problem as

$$\dot{u} = i\frac{\partial H(u,\bar{u})}{\partial \bar{u}}$$

and if we think of u as the infinite dimension vector given by its Fourier coefficients $(\hat{u}(k))_{k\in\mathbb{Z}^n} = (a_k, b_k)_{k\in\mathbb{Z}^n}$, then this becomes an infinite dimension Hamiltonian system.

Lebowitz, Rose and **Speer** considered the Gibbs measure *formally* given by

$$\text{“}d\mu = \exp\left(-\beta H(u)\right)\prod_{x\in\mathbb{T}} du(x)\text{”}$$

for $\beta > 0$ and showed that μ is a well-defined probability measure on $H^s(\mathbb{T})$ for any $s < \frac{1}{2}$.

The Guassian Measure

How do we make sense of the Gibbs measure introduced above? We need to go through the Gaussian measure. Note that the quantity

$$H(u) + \frac{1}{2}\int |u|^2(x)\,dx$$

is conserved. Then the best way to make sense of the Gibbs measure μ is by writing it as

$$d\mu = \exp\left(\frac{1}{6}\int |u|^6\,dx\right)\exp\left(-\frac{1}{2}\int (|u_x|^2 + |u|^2)\,dx\right)\prod_{x\in\mathbb{T}} du(x).$$

In this expression

$$d\rho = \exp\left(-\frac{1}{2}\int (|u_x|^2 + |u|^2)\,dx\right)\prod_{x\in\mathbb{T}} du(x)$$

is the Gaussian measure and

$$\frac{d\mu}{d\rho} = \exp\left(\frac{1}{6}\int |u|^6\,dx\right),$$

corresponding to the nonlinear term of the Hamiltonian, is understood as the Radon-Nikodym derivative of μ with respect to ρ.

Invariance of the Gibbs Measure and Almost Surely Global Well-posedness

Theorem

Consider the Cauchy problem

(5.1)
$$\begin{cases} (i\partial_t + \Delta)u = -|u|^4 u \\ u(0,x) = u_0(x), \ \ \textit{where } x \in \mathbb{T}. \end{cases}$$

The Gibbs measure μ is well defined in H^s, $0 < s < 1/2$ and there exists $\Omega \subset H^s$ such that $\mu(\Omega) = 1$ and (5.1) is globally well-posed in Ω. Moreover μ is **invariant**.

Remark

If one considers (5.1) in the focusing case, then the theorem above holds if one imposes the restriction that the mass is small.

For almost surely global well-posedness results see **Burq-Tzevtkov**, **Oh**, **Oh-Nahmod-Rey-Bellet-S** and **Thomann-Tzevtkov**.

The Non-Squeezing Theorem

We start again with a finite dimension Hamiltonian system. We recall a version of **Gromov's** famous theorem:

Theorem (Finite Dimension Non-squeezing)

Assume that Φ_t is the flow generated by a finite dimension Hamiltonian system as recalled above. Fix $y_0 \in \mathbb{R}^{2k}$ and let $B_r(y_0)$ be the ball in $\mathbb{R}^{2k}$ centered at y_0 and radius r. If

$$C_R(z_0) := \{y = (q_1, \ldots, q_k, p_1, \ldots, p_k) \in \mathbb{R}^{2k} / |q_i - z_0| \leq R\},$$

is a cylinder of radius R, and

$$\Phi_t(B_r(y_0)) \subset C_R(z_0),$$

it must be that $r \leq R$.

Can one generalize this theorem to the infinite dimensional setting given by a periodic dispersive equation?

The infinite dimension Non-squeezing Theorem

Above we recalled how a finite dimensional Hamiltonian flow Φ_t cannot squeeze a ball into a cylinder with a smaller radius. Generalizing this kind of result in infinite dimensions has been a long time project of **Kuksin** who proved, roughly speaking, that compact perturbations of certain linear dispersive equations do indeed satisfy the non-squeezing theorem.

We consider the Cauchy problem

$$\begin{cases} (i\partial_t + \Delta)u = -|u|^2 u \\ u(0,x) = u_0(x), \text{ where } x \in \mathbb{T}. \end{cases}$$

Also in this case, using Strichartz estimates and the conservation of mass ne can prove global well-posedness for data in L^2, see **Bourgain**. Hence we can define a global flow map

$$\Phi(t)u_0 := u(x,t).$$

It is easy to show that the L^2 space equipped with the form

$$\omega(f,g) = \langle if, g\rangle_{L^2}$$

is a symplectic space for the cubic, defocusing NLS equation on $\mathbb{T}$ and its global flow $\Phi(t)$ is a symplectomorphism.

The cubic, periodic, defocusing nonlinear Schrödingier Cauchy problem introduced above is not a compact linear perturbation, hence it is not covered by Kuksin's work. Nevertheless **Bourgain** proved the following theorem:

Theorem (Infinite Dimension Non-squeezing)

Assume that Φ_t is the flow generated by the cubic, periodic, defocusing NLS equation in L^2. If we identify L^2 with l^2 via Fourier transform, we let $B_r(y_0)$ be the ball in l^2 centered at $y_0 \in l^2$ and radius r,

$$C_R(z_0) := \{(a_n) \in l^2 / |a_i - z_0| \leq R\}$$

a cylinder of radius R and

$$\Phi_t(B_r(y_0)) \subset C_R(z_0),$$

at some time t, then it must be that $r \leq R$.

Idea of the Proof

The proof of this theorem is based on the following steps

- Use the projection operator P_N to project the Cauchy problem onto a finite dimension Hamiltonian system.
- Use Gromov's Theorem.
- Show that the flow $\Phi_N(t)$ of the projected problem approximates well the flow $\Phi(t)$ of the original problem.

The third item is the most difficult to prove. The tools used are strong multilinear estimates based on the Strichartz estimates.

Remark

Unfortunately Bourgain's argument may not work for other kinds of dispersive equations. For example for the KdV problem, the lemma in Bourgain's work that gives the good approximation of the flow $\Phi(t)$ by $\Phi_N(t)$ does not hold. This has to do with the number of interacting waves in the nonlinearity. For the KdV problem one can still prove the non-squeezing theorem holds, but the proof was indirect and it had to go through the Miura transformation, see **Colliander-Keel-S-Takaoka- Tao**.

Some Open Problems

- Understand Strichartz estimates for irrational tori.
- Improve theorems on weak turbulence.
- Understand better ergodic structures associated to infinite dimensions Hamiltonian flows.
- Prove well-posedness results using a more probabilistic approach, i.e. by taking appropriate randomized initial data.
- Find more robust arguments to understand the symplectic structures associated to certain dispersive flows.