Self-similarity of the turbulence mixing with a constant macroscale gradient

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Shearless turbulence mixing.
Part 1: numerical experiments

Numerical simulations (DNS and LES) have been carried out with

- Energy ratio $1 < \mathcal{E} \leq 58.3$
- Scale ratio $0.6 \leq \mathcal{L} \leq 2.1$
- Reynolds number: $Re_\lambda \approx 45$ (DNS, LES) and $Re_\lambda \approx 450$ (LES)
- Numerical method: Fourier-Galerkin pseudospectral on a $2\pi \times 2\pi \times 4\pi$ parallelepiped
  Resolution: DNS = $128^2 \times 256$, LES = $32^2 \times 64$
- Initial conditions: two homogeneous turbulent fields ...
flow parameters

• Homogeneous decaying turbulent fields

\[ E = A(t + t_0)^{-n} \]  

\[ n_1 \approx n_2: \text{energy and scale ratio remain constant} \]

\[ \frac{\mathcal{L}(t)}{\mathcal{L}(0)} = \left( 1 + \frac{t}{t_{01}} \right)^{1 - \frac{n_1}{2}} \left( 1 + \frac{t}{t_{02}} \right)^{-1 + \frac{n_2}{2}} \]  

\[ \frac{\mathcal{E}(t)}{\mathcal{E}(0)} = \left( 1 + \frac{t}{t_{02}} \right)^{n_2} \left( 1 + \frac{t}{t_{01}} \right)^{-n_1} \]  

• \( Re_\lambda \approx 45 \)
\[
\mathcal{E} = \frac{E_1}{E_2} \quad \nabla_s(E/E_1) \quad \mathcal{L} = \frac{\ell_1}{\ell_2} \quad \nabla_s(\ell/\ell_1) \quad Re_{\lambda_1} \quad n_1 \quad n_2 \quad \frac{x_s - x_c}{\Delta} \quad \frac{x_k - x_c}{\Delta}
\]

**DNS:**
- A 6.7 0.425 1.0 (1.6) 0.0 45.4 1.22 1.16 0.42 0.62
- B 6.6 0.424 0.6 (0.5) 0.33 45.4 1.22 1.32 0.36 0.45
- C1-s 6.6 0.424 1.5 (1.7) -0.17 45.4 1.22 1.39 0.63 0.84
- C1-c 6.5 0.423 1.5 (1.8) -0.17 45.4 1.22 1.37 0.63 0.84
- C2 6.5 0.423 2.1 (2.7) -0.26 45.4 1.22 1.56 0.79 0.98

**LES:**
- a1 1.43 0.149 1.0 0.0 45 1.25 1.16 0.17 0.30
- a2 6.7 0.421 1.0 0.0 45 1.25 1.15 0.54 0.75
- a3 12.1 0.454 1.0 0.0 45 1.23 1.15 0.65 0.86
- a4 24.0 0.474 1.0 0.0 45 1.22 1.13 0.82 0.99
- a5 58.1 0.485 1.0 0.0 45 1.22 1.13 1.07 1.23
- b1 24.0 0.474 0.53 0.0 45 1.24 1.35 0.72 0.86
- b2 58.0 0.485 0.38 0.0 45 1.20 1.11 0.80 0.91
- V&W - bars 6.2 (2.4) 78.1 1.22 1.39 0.30 0.30
- V&W - plate 6.3 (2.2) 44.5 1.43 1.25 0.63 0.81
- Briggs et al. 7.5 1.0(1.7) 40.3 1.55 1.35 0.38 0.51
- Knaepen et al. 6.27 (2.2) 69.0 1.30 1.10 0.77 1.06

**Tabella 1:** Flow parameters. E=kinetic energy, \(\ell=\)integral scale (eq.2.1), \(Re_{\lambda}=\)Taylor’s microscale Reynolds number, \(\nabla_s = \partial / \partial (x/\Delta)\)=gradient normalized with the mixing layer thickness \(\Delta\), \(n=\)exponent of the energy decay, \(x_s\) and \(x_k\) are the positions of the maxima of skewness and kurtosis. Index 1 refers to the high energy region, index 2 to the low energy region. In the fourth column, the data in parenthesis refer to scales computed through eq. 2.2.
Energy similarity profiles

(a) Region 1
Region 2
C2
C1
C
A
B

(b) $E(\kappa)$

$\log_{10}(\kappa)$
$10^{-5}$
$10^{-4}$
$10^{-3}$
$10^{-2}$
$10^{-1}$
$0.0$
$0.2$
$0.4$
$0.6$
$0.8$
$1.0$

$\frac{(E-E_2)}{(E_1-E_2)}$ 

$(x-x_c)/\Delta$
Higher order moments: skewness and kurtosis profiles

\[ S = \frac{u^3}{u^{2.2}} \quad K = \frac{u^4}{u^{2.2}} \]

Case A: \( \mathcal{E} = 6.7, \mathcal{L} = 1 \), the two fields have same integral scale.
Case B: \( \mathcal{E} = 6.6, \mathcal{L} = 0.6 \), the gradients of energy and scales are opposite: larger scale turbulence has less energy
Case C: \( \mathcal{E} = 6.5, \mathcal{L} = 1.5 \): the gradients of energy and scales have the same sign: larger scale turbulence has more energy.
Penetration - position of maximum of skewness/kurtosis

\[ \frac{(x_s + x_k) / 2 - x_c}{\Delta} \]

\( E_1 / E_2 \)

(a)

\( l_1 / l_2 \)

\( <1 =1 >1 \)

- DNS
- LES, IAM model, \( Re_\lambda = 45 \)
- LES, IAM model, \( Re_\lambda = 450 \)
- Knaepen et al. (2004)
- Pope & Haworth (1987)
Penetration with $\mathcal{L} = 1$

Scaling law (energy ratio):

$$\frac{\eta_s + \eta_k}{2} \sim a \left( \frac{E_1}{E_2} - 1 \right)^b$$

$a \simeq 0.36$, $b \simeq 0.298$

Scaling law (energy gradient):

$$\nabla^* \left( \frac{E}{E_1} \right) \simeq \frac{1 - \mathcal{E}^{-1}}{2}$$

$$\frac{\eta_s + \eta_k}{2} \sim a \left( \frac{2\nabla^* \left( \frac{E}{E_1} \right)}{1 - 2\nabla^* \left( \frac{E}{E_1} \right)} \right)^b$$
• bla bla
• bla
**Part 2: similarity solutions**

To carry out the similarity analysis, we considered the second order moment equations for single-point velocity correlations

\[
\begin{align*}
\partial_t \overline{u'^2} + \partial_x \overline{u'^3} &= -2\rho^{-1} \partial_x \overline{pu} + 2\rho^{-1} \overline{p\partial_x u} - 2\varepsilon_u + \nu \partial_x^2 \overline{u'^2} \quad (4) \\
\partial_t \overline{v'^2_1} + \partial_x \overline{v'^2_1 u} &= 2\rho^{-1} \overline{p\partial_{y_1} v_1} - 2\varepsilon_{v_1} + \nu \partial_x^2 \overline{v'^2_1} \quad (5) \\
\partial_t \overline{v'^2_2} + \partial_x \overline{v'^2_2 u} &= 2\rho^{-1} \overline{p\partial_{y_2} v_2} - 2\varepsilon_{v_2} + \nu \partial_x^2 \overline{v'^2_2} \quad (6)
\end{align*}
\]

where:

- \( u \) is the velocity fluctuation in the inhomogeneous direction \( x \),
- \( v_1, v_2 \) are the velocity fluctuations in the plane \( (y_1, y_2) \) normal to \( x \),
- \( \varepsilon \) is the dissipation.
boundary conditions:

outside the mixing, turbulence is homogeneous and isotropic:

• For \( x \to -\infty \) (high-energy turbulence):

\[
\overline{u^2} = \overline{v_1^2} = \overline{v_2^2} = \frac{2}{3}E_1(t)
\]

\[
\overline{pu} = \overline{u^3} = \overline{v_1^2u} = \overline{v_2^2u} = 0
\]

• For \( x \to +\infty \) (low-energy turbulence):

\[
\overline{u^2} = \overline{v_1^2} = \overline{v_2^2} = \frac{2}{3}E_2(t)
\]

\[
\overline{pu} = \overline{u^3} = \overline{v_1^2u} = \overline{v_2^2u} = 0
\]
Hypothesis and simplifications

• The two homogenous turbulences decay in the same way, thus

\[ E_1(t) = A_1(t + t_0)^{-n_1}, \quad E_2(t) = A_2(t + t_0)^{-n_2} \]

the exponents \( n_1, n_2 \) are close each other (see experiments, Tordella & Iovieno, 2004). Here, we suppose \( n_1 = n_2 = n = 1 \), a value which corresponds to \( R_\lambda \gg 1 \) (Batchelor & Townsend, 1948).

• In the absence of energy production, the pressure-velocity correlation has been shown to be approximately proportional to the convective fluctuation transport (Yoshizawa, 1982, 2002)

\[ -\rho^{-1} \overline{pu} = a \frac{\overline{u^3} + 2\overline{v_1^2}u}{2}, \quad a \approx 0.10, \]

• Single-point second order moments are almost isotropic through the mixing:

\[ \overline{u^2} \approx \overline{v_i^2} \]
hypothesys (2) and (3) implies that pressure-velocity correlations can be expressed in terms of third order velocity correlations:

\[
\bar{u}^2 \simeq v_i^2, \quad \Rightarrow \quad \bar{u}^3 - v_i^2u \simeq 2\rho^{-1}p\partial_xu
\]

then

\[-\rho^{-1}p\bar{u} = \alpha\bar{u}^3, \quad \alpha = \frac{3a}{1 + 2a} \approx 0.25. \quad (7)\]
Similarity hypothesis

The moment distributions in the above problem are determined by

• the coordinates \( x, t \)
• the energy \( E_1(t), E_2(t) \) of the two mixing turbulences.
• the scales \( \ell_1(t), \ell_2(t) \) of the two mixing turbulences.

Thus, through dimensional analysis,

\[
\overline{u^2} = E_1^\frac{1}{2} \varphi_2(\eta, R_{\ell_1}, \vartheta_1, \mathcal{E}, \mathcal{L}) \tag{8}
\]

\[
\overline{u^3} = E_1 \varphi_{uuu}(\eta, R_{\ell_1}, \vartheta_1, \mathcal{E}, \mathcal{L}) \tag{9}
\]

\[
\varepsilon_u = E_1^\frac{3}{2} \ell_1^{-1} \varphi_{\varepsilon_u}(\eta, R_{\ell_1}, \vartheta_1, \mathcal{E}, \mathcal{L}), \tag{10}
\]

where:

\( \eta = x/\Delta(t) \), \( \Delta(t) \) is the mixing layer thickness, \( R_{\ell_1} = E_1^\frac{1}{2}(t)\ell_1(t)/\nu \),

\( \vartheta_1 = tE_1^\frac{1}{2}(t)/\ell_1(t) \) \( \mathcal{E} = E_1(t)/E_2(t) \), \( \mathcal{L} = \ell_1(t)/\ell_2(t) \)
The high Reynolds number algebraic decay \((n = 1)\), implies:

\[
\mathcal{E} = \text{const} = \frac{E_1(0)}{E_2(0)} \tag{11}
\]

\[
\mathcal{L} = \text{const} = \frac{\ell_1(0)}{\ell_2(0)} \tag{12}
\]

\[
\vartheta_1 = \text{const} = \frac{n}{f(R_{\lambda_1})} \tag{13}
\]

\[
R_{\ell_1} \propto t^{1-n} = \text{const} \tag{14}
\]

\[\Rightarrow \eta \text{ is the only similarity variable function of } x, t.\]
⇒ similarity conditions:

After having introduced the similarity relations into equation ??, all coefficient must be independent from $x, t$, so that

$$\Delta(t) \propto \ell_1(t)$$

⇒ similarity equation:

$$-\frac{1}{2}\eta \frac{\partial \varphi_{uu}}{\partial \eta} + \frac{1}{f(R_{\lambda_1})}(1 - 2\alpha) \frac{\partial \varphi_{uuu}}{\partial \eta} + \frac{\nu}{Af(R_{\lambda_1})^2} \frac{\partial^2 \varphi_{uu}}{\partial \eta^2} =$$

$$\varphi_{uu} - \frac{2}{f(R_{\lambda_1})}\varphi_{\varepsilon u}$$

(15)
⇒ third-order moments can be expressed a function of second order moments: the skewness is then

\[ \varphi_{uuu} = \frac{1}{(1 - 2\alpha)} \left[ \frac{f}{2} \int_{-\infty}^{\eta} \eta \frac{\partial \varphi_{uu}}{\partial \eta} d\eta + \frac{\nu}{A_1 f} \frac{\partial \varphi_{uu}}{\partial \eta} + \beta f \frac{\partial^3 \varphi_{uu}}{\partial \eta^3} \right] \]  \hspace{1cm} (16)

\[ S = \frac{\varphi_{uu}^3}{(1 - 2\alpha)} \left[ \frac{f}{2} \int_{-\infty}^{\eta} \eta \frac{\partial \varphi_{uu}}{\partial \eta} d\eta + \frac{\nu}{A_1 f} \frac{\partial \varphi_{uu}}{\partial \eta} + \beta f \frac{\partial^3 \varphi_{uu}}{\partial \eta^3} \right] \]  \hspace{1cm} (17)
Numerical experiments suggest the following fit for second-order moments (see also Veeravalli & Wahrhaft, *JFM 1989*)

\[
\varphi_{uu} = \frac{1 + \mathcal{E}^{-1}}{2} - \frac{1 - \mathcal{E}^{-1}}{2} \text{erf}(\eta) \quad (18)
\]

so that we can analytically carry out integrations to compute the velocity skewness

\[
S = \frac{1 - \mathcal{E}^{-1}}{\sqrt{\pi}} e^{-\eta^2} \left[ \frac{f}{4(1 - 2\alpha)} \left( 1 - \frac{4\nu}{A_1 f^2} \right) + 2\beta(1 - 2\eta) \right] \times \left[ \frac{1 + \mathcal{E}^{-1}}{2} - \frac{1 - \mathcal{E}^{-1}}{2} \text{erf}(\eta) \right]^{-\frac{3}{2}} \quad (19)
\]
Normalized energy and skewness distributions; $E = 6.7$ and $L = 1$. 

![Diagram (a)](image1.png)

![Diagram (b)](image2.png)
Position of the maximum of skewness

\( \frac{(x_s - x_c)}{\Delta} \)

(a)

\( E_1 / E_2 \)

\( l_1 / l_2 \)

\(< 1 = 1 > 1\)

- DNS
- LES, IAM model

Similarity solution
Conclusions

The intermediate asymptotics of turbulence mixings in the absence of the production of turbulent kinetic energy has been considered.

- A similarity stage of the decay of shearless turbulence mixing always exist, even if it still depends on initial flow parameters.
- If $E$ is far from unity, the mixing is very intermittent.
- Intermittency smoothly varies when passing through $L = 1$:
  - It increases when $L > 1$, it is reduced when $L < 1$
- When $L = 1$, the intermittency increases with the energy ratio $E$ with a scaling exponent that is almost equal to 0.29.
- A similarity decay of shearless mixing is consistent with single-point correlation equations