Hydrodynamic linear stability of the 2D bluff-body wake through modal analysis and initial-value problem formulation

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Outline

1. Introduction

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2. Physical Problem
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5. Transient and Long-Term Behavior of Small 3D Perturbations
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- Hydrodynamics stability is important in different fields (aerodynamics, oceanography, atmospheric sciences, etc).
Linear stability analysis of the 2D bluff-body wake

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- **Two-dimensional wake past a circular cylinder**
  - Circular cylinder is the quintessential bluff-body;
  - Important prototype of free shear flow for the study and applications in fluid mechanics.
Modal analysis vs Initial-value problem

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  - Flow asymptotically stable or unstable;
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- **Initial-value problem**
  - Temporal evolution of arbitrary disturbances;
  - Importance of the transient growth (e.g. *by-pass transition*);
  - Aim to understand the cause of any possible instability in terms of the underlying physics.
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  ⇒ Steady, incompressible and viscous;
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Normal Mode Theory

The linearized perturbative equation in terms of stream function \( \psi(x, y, t) \) is

\[
\partial_t \nabla^2 \psi + (\partial_x \nabla^2 \psi) \psi_y + \psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \psi) \psi_x - \psi_x \partial_y \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi
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**Absolute instability**: \( r_0 > 0 \), \( \partial \sigma_0 / \partial h_0 = 0 \) for at least one mode.
Stability analysis through multiscale approach

- Slow variables: \( x_1 = \epsilon x, \ t_1 = \epsilon t, \ \epsilon = 1/Re. \)
Stability analysis through multiscale approach

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- **Hypothesis:** $\psi(x, y, t)$ and $\Psi(x, y, t)$ are expansions in terms of $\epsilon$:
  
  $$(\text{ODE dependent on } \varphi_0) + \epsilon (\text{ODE dependent on } \varphi_0, \varphi_1) + O(\epsilon^2)$$
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- **Hypothesis:** \( \psi(x, y, t) \) and \( \Psi(x, y, t) \) are expansions in terms of \( \epsilon \):
  - (ODE dependent on \( \varphi_0 \)) + \( \epsilon \) (ODE dependent on \( \varphi_0, \varphi_1 \)) + \( O(\epsilon^2) \)
- **Order zero:** homogeneous Orr-Sommerfeld equation
  \[ A \varphi_0 = \sigma_0 B \varphi_0 \]
  \[ A = (\partial_y^2 - h_0^2)^2 - i h_0 Re[u_0(\partial_y^2 - h_0^2) - \partial_y^2 u_0] \]
  \[ \varphi_0 \rightarrow 0, \ |y| \rightarrow \infty \]
  \[ \partial_y \varphi_0 \rightarrow 0, \ |y| \rightarrow \infty \]
  \[ \Rightarrow \text{eigenfunctions} \ \varphi_0 \text{ and a discrete set of eigenvalues} \ \sigma_{0n}. \]
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- **Hypothesis:** \( \psi(x, y, t) \) and \( \Psi(x, y, t) \) are expansions in terms of \( \epsilon: \) 
  
  \[(\text{ODE dependent on } \varphi_0) + \epsilon (\text{ODE dependent on } \varphi_0, \ \varphi_1) + O(\epsilon^2)\]

- **Order zero:** homogeneous Orr-Sommerfeld equation

  \[
  \mathcal{A} \varphi_0 = \sigma_0 \mathcal{B} \varphi_0 \quad \mathcal{A} = (\partial_y^2 - h_0^2)^2 - ih_0 \text{Re}[u_0(\partial_y^2 - h_0^2) - \partial_y^2 u_0] \\
  \varphi_0 \rightarrow 0, \ |y| \rightarrow \infty \quad \mathcal{B} = -i\text{Re}(\partial_y^2 - h_0^2) \\
  \partial_y \varphi_0 \rightarrow 0, \ |y| \rightarrow \infty \\
  \Rightarrow \text{eigenfunctions } \varphi_0 \text{ and a discrete set of eigenvalues } \sigma_{0n}.
  
- **First order:** Non homogeneous Orr-Sommerfeld equation

  \[
  \mathcal{A} \varphi_1 = \sigma_0 \mathcal{B} \varphi_1 + \mathcal{M} \varphi_0 \quad \mathcal{M} = \left[ \text{Re}(2h_0 \sigma_0 - 3h_0^2 u_0 - \partial_y^2 u_0) + 4ih_0^3 \right] \partial_{x_1} \\
  \varphi_1 \rightarrow 0, \ |y| \rightarrow \infty \quad + (\text{Re}u_0 - 4ih_0) \partial_{x_1}^3 \partial_{yy} - \text{Re}v_1 (\partial_y^3 - h_0^2 \partial_y) + \text{Re} \partial_y^2 v_1 \partial_y \\
  \partial_y \varphi_1 \rightarrow 0, \ |y| \rightarrow \infty \quad + ih_0 \text{Re} \left[ u_1 (\partial_y^2 - h_0^2) - \partial_y^2 u_1 \right] + \text{Re} (\partial_y^2 - h_0^2) \partial_{t_1}
  
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Perturbative hypothesis: saddle point sequence

For fixed values of $x$ and $Re$, the saddle points $(h_{0s}, \sigma_{0s})$ of the dispersion relation $\sigma_0 = \sigma_0(h_0, x, Re)$ satisfy $\frac{\partial \sigma_0}{\partial h_0} = 0$;
Perturbative hypothesis: saddle point sequence

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$Re = 35, x = 4$. Level curves, $\omega_0 =$ const (thin curves), $r_0 =$ const (thick curves).
Perturbative hypothesis: saddle point sequence

$Re = 35, \ x = 4. \ \omega_0(k_0, s_0), \ r_0(k_0, s_0)$. 
Instability Characteristics

(a) $k_0$, $k$

(b) $s_0$, $s$

(c) $\omega_0$, $\omega$

(d) $r_0$, $r$

Re=35 - solid line
Re=50 - dashed line
Re=100 - dotted line
Global Pulsation

- Comparison between present solution (accuracy $\Delta \omega = 0.05$), Zebib’s numerical study (1987), Pier’s direct numerical simulations (2002), Williamson’s experimental results (1988).

\[
\begin{array}{c}
\text{Re} \\
30 \quad 45 \quad 60 \quad 75 \quad 90 \quad 105 \\
\end{array}
\]

\[
\begin{array}{c}
\omega \\
0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1 \\
\end{array}
\]

Defect of the volumetric flow rate $D$:

$$D(x) = \int_{-\infty}^{+\infty} (1 - U(x, y)) dy$$

The first $R_{cr}$ as a possible measure of the entrainment length.

**Velocity Flow Rate Defect and Entrainment**
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- **Defect of the volumetric flow rate** $D$:

$$D(x) = \int_{-\infty}^{+\infty} (1 - U(x, y)) dy$$

- **Entrainment** $E$ takes into account the variation of the defect of the volumetric flow rate in the streamwise direction:

$$E(x) = \left| \frac{dD(x)}{dx} \right|$$

Results

The first $R_{cr}$ as a possible measure of the entrainment length
Formulation

- Linear three-dimensional perturbative equations in terms of velocity and vorticity (Criminale & Drazin, 1990);
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- Base flow parametric in \(x\) and \(Re \Rightarrow U(y; x_0, Re)\);
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- Base flow parametric in $x$ and $Re \Rightarrow U(y; x_0, Re)$;
- Laplace-Fourier transform in $x$ and $z$ directions, $\alpha$ complex, $\gamma$ real;
Formulation

- Linear three-dimensional perturbative equations in terms of velocity and vorticity (Criminale & Drazin, 1990);
- Base flow parametric in $x$ and $Re \Rightarrow U(y; x_0, Re)$;
- Laplace-Fourier transform in $x$ and $z$ directions, $\alpha$ complex, $\gamma$ real;

\[\begin{align*}
\gamma & = \text{transversal wavenumber} \\
\alpha_r & = \text{longitudinal wavenumber} \\
\phi & = \text{angle of obliquity} \\
k & = \text{polar wavenumber} \\
\alpha_i & = \text{spatial damping rate}
\end{align*}\]
Perturbative equations

- Perturbative linearized system:

\[
\frac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i) \hat{v} = \hat{\Gamma}
\]

\[
\frac{\partial \hat{\Gamma}}{\partial t} = (i\alpha_r - \alpha_i) \left( \frac{d^2 U}{dy^2} \hat{v} - U\hat{\Gamma} \right) + \frac{1}{Re} \left[ \frac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i) \hat{\Gamma} \right]
\]

\[
\frac{\partial \hat{\omega}_y}{\partial t} = -(i\alpha_r - \alpha_i) U \hat{\omega}_y - i\gamma \frac{dU}{dy} \hat{v} + \frac{1}{Re} \left[ \frac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i) \hat{\omega}_y \right]
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Perturbative equations

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\]

The transversal velocity and vorticity components are \(\hat{\nu}\) and \(\hat{\omega}_y\) respectively, \(\hat{\Gamma}\) is defined as \(\hat{\Gamma} = \partial_x\hat{\omega}_z - \partial_z\hat{\omega}_x\).
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- Initial conditions:
  - \(\hat{\omega}_y(0, y) = 0\);
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- Initial conditions:
  - \(\hat{\omega}_y(0, y) = 0\);  \\
  - \(\hat{\Gamma}(0, y) = e^{-y^2}\sin(y)\) or \(\hat{\Gamma}(0, y) = e^{-y^2}\cos(y)\);
Perturbative equations

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  The transversal velocity and vorticity components are \( \hat{\nu} \) and \( \hat{\omega}_y \) respectively, \( \hat{\Gamma} \) is defined as \( \tilde{\Gamma} = \partial_x \hat{\omega}_z - \partial_z \hat{\omega}_x \).

- Initial conditions:
  - \( \hat{\omega}_y(0, y) = 0; \)
  - \( \hat{\Gamma}(0, y) = e^{-y^2} \sin(y) \) or \( \hat{\Gamma}(0, y) = e^{-y^2} \cos(y); \)

- Boundary conditions: \( (\hat{u}, \hat{v}, \hat{w}) \rightarrow 0 \) as \( y \rightarrow \infty \).
Measure of the Growth

- Kinetic energy density $e$:

$$
e(t; \alpha, \gamma) = \frac{1}{2} \frac{1}{2y_d} \int_{-y_d}^{y_d} (|\hat{u}|^2 + |\hat{v}|^2 + |\hat{w}|^2) dy$$

$$= \frac{1}{2} \frac{1}{2y_d} \frac{1}{|\alpha^2 + \gamma^2|} \int_{-y_d}^{y_d} \left( |\frac{\partial \hat{v}}{\partial y}|^2 + |\alpha^2 + \gamma^2||\hat{v}|^2 + |\hat{\omega}_y|^2 \right) dy$$
Measure of the Growth

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\]

\[
= \frac{1}{2} \frac{1}{2} \frac{1}{2y_d |\alpha^2 + \gamma^2|} \int_{-y_d}^{+y_d} (|\hat{\nu}|^2 + |\alpha^2 + \gamma^2||\hat{v}|^2 + |\hat{\omega}_y|^2) dy
\]

- **Amplification factor \( G \):**

\[
G(t; \alpha, \gamma) = \frac{e(t; \alpha, \gamma)}{e(t = 0; \alpha, \gamma)}
\]
Measure of the Growth

- **Temporal growth rate** $r$ (*Lasseigne et al.*, 1999):

  $$
  r(t; \alpha, \gamma) = \frac{\log|e(t; \alpha, \gamma)|}{2t}, \quad t > 0
  $$
Measure of the Growth

- **Temporal growth rate** $r$ \((\text{Lasseigne et al., 1999})\):

  \[
  r(t; \alpha, \gamma) = \frac{\log |e(t; \alpha, \gamma)|}{2t}, \quad t > 0
  \]

- **Angular frequency (pulsation)** $\omega$ \((\text{Whitham, 1974})\):

  \[
  \omega(t; \alpha, \gamma) = \frac{d\varphi(t)}{dt}, \quad \varphi \text{ time phase}
  \]
Effect of $\alpha_i$ and $k$

Effect of the symmetry of the perturbation

(a) $x_0=10$ (intermediate)  
--- $x_0=50$ (far)

$\tau_{\text{far}} = 100$

Re=100

$k=0.6$

$\alpha_i=0.02$

$\phi = \pi/4$

asymmetric input

(b) $x_0=10$ (intermediate)  
--- $x_0=50$ (far)

Re=100

$k=0.6$

$\alpha_i=0.02$

$\phi = \pi/4$

symmetric input

(c) $k=0.6$  
--- $\alpha_i=0.02$  
--- $\phi = \pi/4$

(d) $x_0=10$  
--- $\text{Re}=100$

--- $\omega$

sym

asym

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Effect of $\phi$

- $\phi=0$
- $\phi=\pi/2$

$\tau_{\text{inter}} = 2.5$

$\alpha_i = 0.02$
$k = 0.7$
$x_0 = 10$
$Re=100$

$\phi=0$
$k = 0.7$
$x_0 = 10$
$Re=100$

$\alpha_i = 0.02$

-asymmetric input-
Comparison with modal analysis and laboratory data
Angular frequency and temporal growth rate, $\alpha_i = 0.05$, $\phi = 0$, $x_0 = 10$.

Full linear problem

- Linearized 3D equations and Laplace-Fourier transform \((x, z)\);
Full linear problem

- Linearized 3D equations and Laplace-Fourier transform \((x, z)\);
- Base flow parametric in \(x\) and \(Re \Rightarrow (U(y; x_0, Re), V(y; x_0, Re))\).
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\[
\frac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2ik\cos(\phi)\alpha_i)\hat{v} = \hat{\Gamma}
\]
\[
\frac{\partial \hat{r}}{\partial t} = G\hat{\Gamma} + H\hat{v} + K\hat{\omega}_y
\]
\[
\frac{\partial \hat{\omega}_y}{\partial t} = L\hat{\omega}_y + M\hat{v}
\]
Full linear problem

- Linearized 3D equations and Laplace-Fourier transform \((x, z)\);
- Base flow parametric in \(x\) and \(Re\) \(\Rightarrow (U(y; x_0, Re), V(y; x_0, Re))\);

\[
\frac{\partial^2 \hat{v}}{\partial y^2} - (k^2 - \alpha_i^2 + 2 i k \cos(\phi) \alpha_i) \hat{v} = \hat{\Gamma} \\
\frac{\partial \hat{f}}{\partial t} = G\hat{\Gamma} + H\hat{v} + K\hat{\omega}_y \\
\frac{\partial \hat{\omega}_y}{\partial t} = L\hat{\omega}_y + M\hat{v}
\]

- \(G = G(y; x_0, k, \phi, \alpha_i, Re), \) and similarly \(H, K, L\) and \(M\), are ordinary differential operators.
Multiple scales hypothesis

- Regular perturbation scheme, $k \ll 1$:

\[ \hat{v} = \hat{v}_0 + k\hat{v}_1 + k^2\hat{v}_2 + \cdots, \]
\[ \hat{\Gamma} = \hat{\Gamma}_0 + k\hat{\Gamma}_1 + k^2\hat{\Gamma}_2 + \cdots, \]
\[ \hat{\omega}_y = \hat{\omega}_{y0} + k\hat{\omega}_{y1} + k^2\hat{\omega}_{y2} + \cdots. \]
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- Regular perturbation scheme, $k \ll 1$:

  \[
  \hat{v} = \hat{v}_0 + k \hat{v}_1 + k^2 \hat{v}_2 + \cdots, \\
  \hat{\Gamma} = \hat{\Gamma}_0 + k \hat{\Gamma}_1 + k^2 \hat{\Gamma}_2 + \cdots, \\
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  \]

- Temporal scales: $t, \tau = kt, T = k^2 t$;
Multiple scales hypothesis

- Regular perturbation scheme, $k \ll 1$:
  \[
  \hat{\nu} = \hat{\nu}_0 + k \hat{\nu}_1 + k^2 \hat{\nu}_2 + \cdots ,
  \hat{\Gamma} = \hat{\Gamma}_0 + k \hat{\Gamma}_1 + k^2 \hat{\Gamma}_2 + \cdots ,
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  \]

- Temporal scales: $t, \tau = kt, T = k^2 t$;
- Spatial scales: $y, Y = ky$. 
**Order O(1)**

\[
\frac{\partial^2 \hat{V}_0}{\partial y^2} + \alpha_i^2 \hat{V}_0 = \hat{\Gamma}_0
\]

\[
\frac{\partial \hat{\Gamma}_0}{\partial t} - G_0 \hat{\Gamma}_0 - H_0 \hat{V}_0 = 0
\]

\[
\frac{\partial \hat{\omega}_y}{\partial t} - L_0 \hat{\omega}_y = 0
\]
Multiple scales equations up to $O(k)$

- **Order $O(1)$**

$$\frac{\partial^2 \hat{v}_0}{\partial y^2} + \alpha_i^2 \hat{v}_0 = \hat{\Gamma}_0$$

$$\frac{\partial \hat{\Gamma}_0}{\partial t} - G_0 \hat{\Gamma}_0 - H_0 \hat{v}_0 = 0$$

$$\frac{\partial \hat{\omega}_y}{\partial t} - L_0 \hat{\omega}_y = 0$$

where $G_0 = G_0(y; x_0, \phi, \alpha_i, Re)$ and similarly for $H_0$ and $L_0$. 

Introduction
Physical Problem
Normal Mode Analysis
Streamwise Entrainment Evolution
Transient and Long-Term Behavior of Small 3D Perturbations
Multiscale analysis for the stability of long 3D waves
Conclusions

Formulation
Comparison between multiscale and full problem results
Multiple scales equations up to $O(k)$

- **Order $O(k)$**

\[
\frac{\partial^2 \hat{v}_1}{\partial y^2} + \alpha_i^2 \hat{v}_1 = -2 \frac{\partial^2 \hat{v}_0}{\partial y \partial Y} + 2i \cos(\phi) \alpha_i \hat{v}_0 + \hat{\Gamma}_1
\]

\[
\frac{\partial \hat{\Gamma}_1}{\partial t} - G_0 \hat{\Gamma}_1 - H_0 \hat{v}_1 = -\frac{\partial \hat{\Gamma}_0}{\partial \tau} + G_1 \hat{\Gamma}_0 + H_1 \hat{v}_0 + K_1 \hat{\omega}_y 0
\]

\[
\frac{\partial \hat{\omega}_y 1}{\partial t} - L_0 \hat{\omega}_y 1 = -\frac{\partial \hat{\omega}_y 0}{\partial \tau} + L_1 \hat{\omega}_y 0 + M_1 \hat{v}_0
\]
Multiple scales equations up to $O(k)$

- Order $O(k)$

\[
\frac{\partial^2 \hat{v}_1}{\partial y^2} + \alpha_i^2 \hat{v}_1 = -2 \frac{\partial^2 \hat{v}_0}{\partial y \partial Y} + 2i \cos(\phi) \alpha_i \hat{v}_0 + \hat{\Gamma}_1
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\]

\[
\frac{\partial \hat{\omega}_y}{\partial t} = - \frac{\partial \hat{\omega}_y}{\partial \tau} + L_1 \hat{\omega}_y + M_1 \hat{v}_0
\]

where $G_1 = G_1(y, Y; x_0, \phi, \alpha_i, Re)$ and similarly for $H_1, K_1, L_1$ and $M_1$. 
Effect of $\alpha_i$ and $k$

Effect of the symmetry of the perturbation

- **Re=100**, **k=0.02**, **ϕ=π/2**
- **x₀=10**, **αᵢ=0.08**

- **G** vs. time **t** for **O(1)** full problem, compared to symmetrical (sym) and asymmetrical (asym) cases.
- **r** vs. time **t** for **O(1)** full problem, compared to symmetrical (sym) and asymmetrical (asym) cases.
Asymptotic state

- Temporal asymptotic values of the angular frequency $\omega$ and the temporal growth rate $r$.

![Graph showing temporal asymptotic values of $\omega$ and $r$ for different cases.](image)
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  - Synthetic perturbation hypothesis (saddle point sequence);
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  - *More difficult handling of the parameters.*
Next Steps

- Energy spectrum of a general pre-unstable large set of *multiple transient three dimensional waves*.
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![Graphical representation of the cross flow boundary layer](image)

- Initial-value problem for compressible flows.

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