Hydrodynamic linear stability of the 2D bluff-body wake through modal analysis and initial-value problem formulation

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  - Important prototype of free shear flow for the study and applications in fluid mechanics.

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  - Importance of the transient growth (e. g. *by-pass transition*);
  - Aim to understand the cause of any possible instability in terms of the underlying physics.

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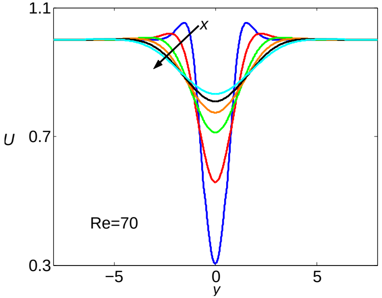

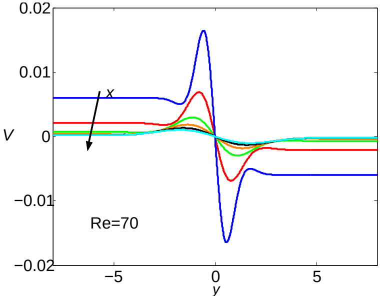

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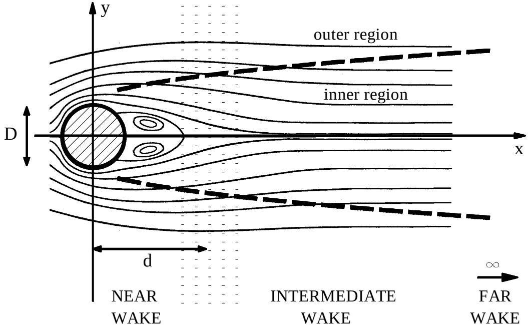

Normal Mode Theory

- The linearized perturbative equation in terms of stream function $\psi(x, y, t)$ is

$$\partial_t \nabla^2 \psi + (\partial_x \nabla^2 \Psi)\psi_y + \Psi_y \partial_x \nabla^2 \psi - (\partial_y \nabla^2 \Psi)\psi_x - \Psi_x \partial_y \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi$$

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- **Absolute instability**: $r_0 > 0$, $\partial \sigma_0 / \partial h_0 = 0$ for at least one mode.

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- **Order zero:** homogeneous Orr-Sommerfeld equation

$$\mathcal{A}\varphi_0 = \sigma_0 \mathcal{B}\varphi_0 \qquad \mathcal{A} = (\partial_y^2 - h_0^2)^2 - ih_0 Re[u_0(\partial_y^2 - h_0^2) - \partial_y^2 u_0]$$

$$\varphi_0 \to 0, |y| \to \infty \qquad \mathcal{B} = -iRe(\partial_y^2 - h_0^2)$$

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$\Rightarrow$ eigenfunctions φ_0 and a discrete set of eigenvalues σ_{0n}.

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- **First order:** Non homogeneous Orr-Sommerfeld equation

$$
\begin{aligned}
&\mathcal{A}\varphi_1 = \sigma_0 \mathcal{B}\varphi_1 + \mathcal{M}\varphi_0 && \mathcal{M} = \left[Re(2h_0\sigma_0 - 3h_0^2 u_0 - \partial_y^2 u_0) + 4ih_0^3\right] \partial_{x_1} \\
&\varphi_1 \to 0, |y| \to \infty && + (Reu_0 - 4ih_0)\partial^3_{x_1 yy} - Rev_1(\partial_y^3 - h_0^2 \partial_y) + Re\partial_y^2 v_1 \partial_y \\
&\partial_y \varphi_1 \to 0, |y| \to \infty && + ih_0 Re\left[u_1(\partial_y^2 - h_0^2) - \partial_y^2 u_1\right] + Re(\partial_y^2 - h_0^2)\partial_{t_1}
\end{aligned}
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Perturbative hypothesis: saddle point sequence

- For fixed values of x and Re, the saddle points (h_{0s}, σ_{0s}) of the dispersion relation $\sigma_0 = \sigma_0(h_0, x, Re)$ satisfy $\partial\sigma_0/\partial h_0 = 0$;

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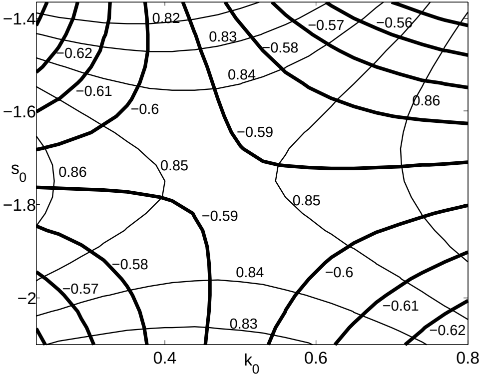

$Re = 35$, $x = 4$. Level curves, ω_0 =const (thin curves), r_0 =const (thick curves).

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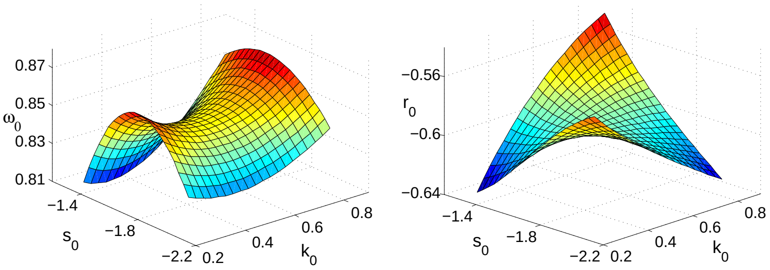
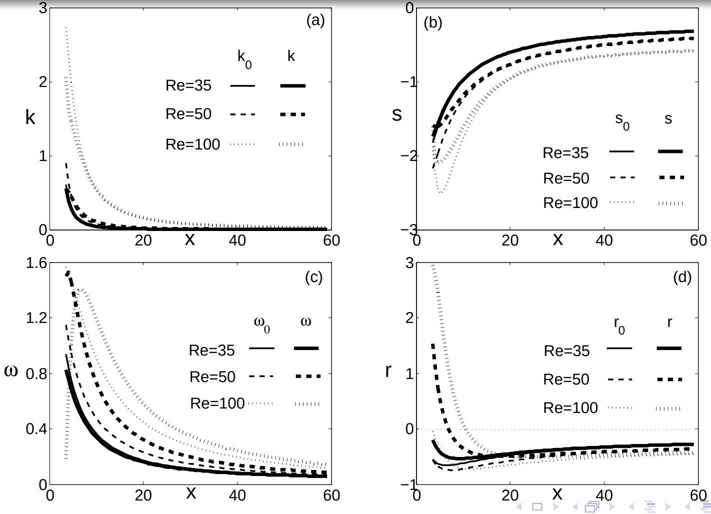


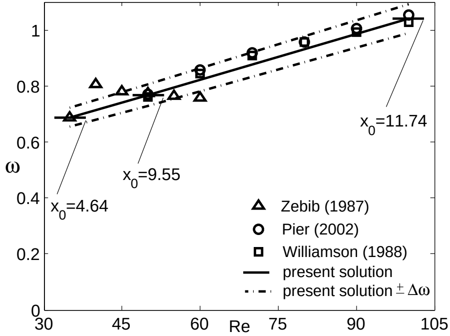
$Re = 35$, $x = 4$. $\omega_0(k_0, s_0)$, $r_0(k_0, s_0)$.

Instability Characteristics



Global Pulsation

- Comparison between present solution (accuracy $\Delta\omega = 0.05$), Zebib's numerical study (1987), Pier's direct numerical simulations (2002), Williamson's experimental results (1988).



Tordella, Scarsoglio & Belan, *Phys. Fluids*, 2006.

Velocity Flow Rate Defect and Entrainment

- Defect of the volumetric flow rate D:

$$D(x) = \int_{-\infty}^{+\infty} (1 - U(x,y))dy$$

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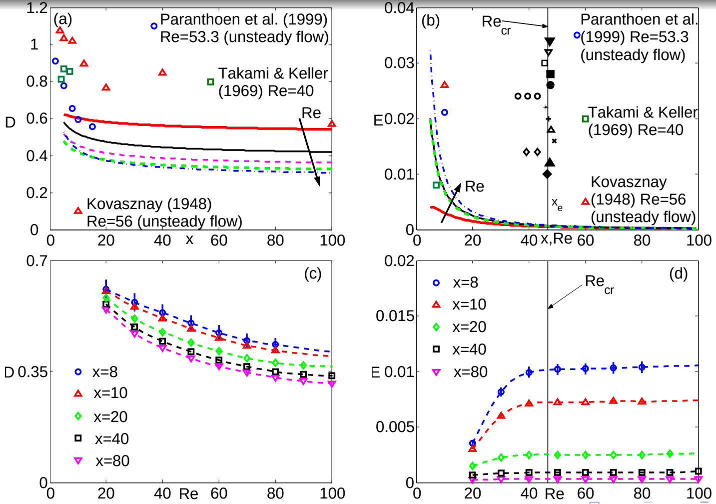
- Entrainment E takes into account the variation of the defect of the volumetric flow rate in the streamwise direction:

$$E(x) = \left| \frac{dD(x)}{dx} \right|$$

Tordella & Scarsoglio, *Phys. Letters A*, 2009.

The first R_{cr} as a possible measure of the entrainment length

Results



Formulation

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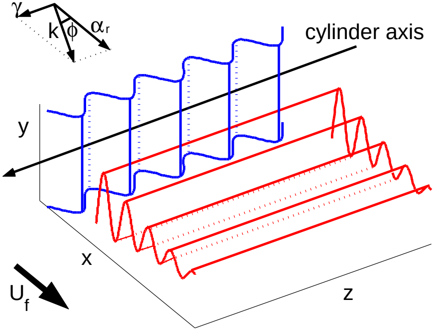

α_r = longitudinal wavenumber
γ = transversal wavenumber
ϕ = angle of obliquity
k = polar wavenumber
α_i = spatial damping rate

Perturbative equations

- Perturbative linearized system:

$$\begin{aligned}
\frac{\partial^2 \hat{v}}{\partial y^2} &- (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{v} = \hat{\Gamma} \\
\frac{\partial \hat{\Gamma}}{\partial t} &= (i\alpha_r - \alpha_i)(\frac{d^2U}{dy^2}\hat{v} - U\hat{\Gamma}) + \frac{1}{Re}[\frac{\partial^2 \hat{\Gamma}}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\Gamma}] \\
\frac{\partial \hat{\omega}_y}{\partial t} &= -(i\alpha_r - \alpha_i)U\hat{\omega}_y - i\gamma\frac{dU}{dy}\hat{v} + \frac{1}{Re}[\frac{\partial^2 \hat{\omega}_y}{\partial y^2} - (k^2 - \alpha_i^2 + 2i\alpha_r\alpha_i)\hat{\omega}_y]
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- Boundary conditions: $(\hat{u}, \hat{v}, \hat{w}) \to 0$ as $y \to \infty$.

Measure of the Growth

- Kinetic energy density e:

$$
\begin{aligned}
e(t;\alpha,\gamma) &= \frac{1}{2}\frac{1}{2y_d}\int_{-y_d}^{+y_d}(|\hat{u}|^2+|\hat{v}|^2+|\hat{w}|^2)dy \\
&= \frac{1}{2}\frac{1}{2y_d}\frac{1}{|\alpha^2+\gamma^2|}\int_{-y_d}^{+y_d}(|\frac{\partial\hat{v}}{\partial y}|^2+|\alpha^2+\gamma^2||\hat{v}|^2+|\hat{\omega}_y|^2)dy
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- Amplification factor G:

$$G(t;\alpha,\gamma)=\frac{e(t;\alpha,\gamma)}{e(t=0;\alpha,\gamma)}$$

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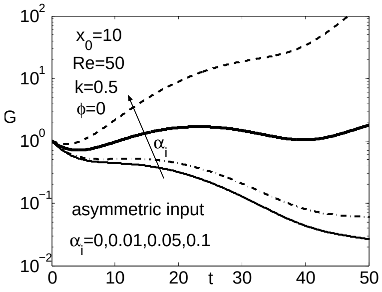
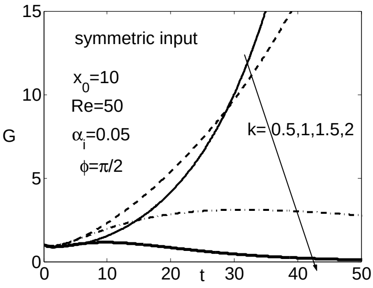
- Angular frequency (pulsation) ω (*Whitham, 1974*):

$$\omega(t;\alpha,\gamma) = \frac{d\varphi(t)}{dt}, \qquad \varphi \text{ time phase}$$

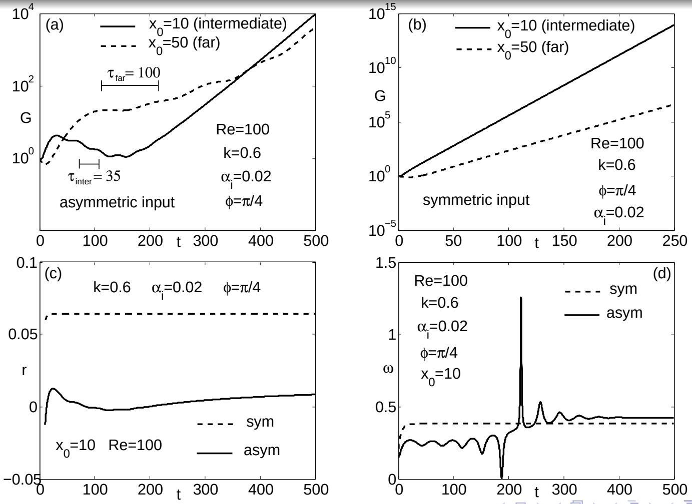
Effect of α_i and k



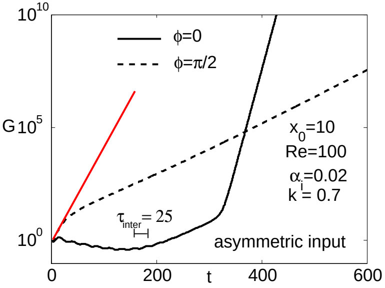


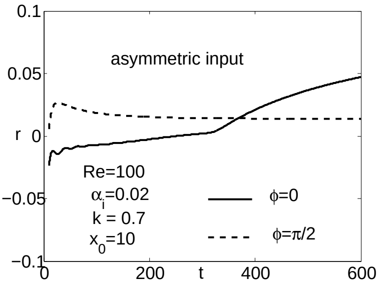
Scarsoglio, Tordella & Criminale, *Stud. Applied Math.*, 2009.

Effect of the symmetry of the perturbation



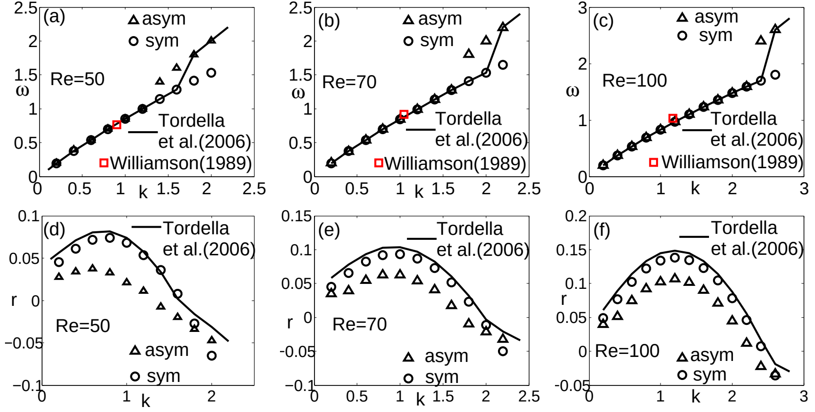
Effect of ϕ





Comparison with modal analysis and laboratory data

Angular frequency and temporal growth rate, $\alpha_i = 0.05$, $\phi = 0$, $x_0 = 10$.



Scarsoglio, Tordella & Criminale, *ETC XII*, 2009.

Full linear problem

- Linearized 3D equations and Laplace-Fourier transform (x, z);

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\frac{\partial^2 \hat{v}}{\partial y^2} & - & (k^2 - \alpha_i^2 + 2ikcos(\phi)\alpha_i)\hat{v} = \hat{\Gamma} \\
\frac{\partial \hat{\Gamma}}{\partial t} & = & G\hat{\Gamma} + H\hat{v} + K\hat{\omega}_y \\
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- $G = G(y; x_0, k, \phi, \alpha_i, Re)$, and similarly H, K, L and M, are ordinary differential operators.

Multiple scales hypothesis

- Regular perturbation scheme, $k \ll 1$:

$$\begin{aligned}
\hat{v} &= \hat{v}_0 + k\hat{v}_1 + k^2\hat{v}_2 + \cdots \ , \\
\hat{\Gamma} &= \hat{\Gamma}_0 + k\hat{\Gamma}_1 + k^2\hat{\Gamma}_2 + \cdots \ , \\
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- Spatial scales: y, $Y = ky$.

Multiple scales equations up to $O(k)$

- **Order O(1)**

$$
\begin{aligned}
\frac{\partial^2 \hat{v}_0}{\partial y^2} + \alpha_i^2 \hat{v}_0 &= \hat{\Gamma}_0 \\
\frac{\partial \hat{\Gamma}_0}{\partial t} - G_0 \hat{\Gamma}_0 - H_0 \hat{v}_0 &= 0 \\
\frac{\partial \hat{\omega}_{y0}}{\partial t} - L_0 \hat{\omega}_{y0} &= 0
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\begin{aligned}
\frac{\partial^2 \hat{v}_1}{\partial y^2} \quad + \quad \alpha_i^2 \hat{v}_1 &= -2\frac{\partial^2 \hat{v}_0}{\partial y \partial Y} + 2icos(\phi)\alpha_i \hat{v}_0 + \hat{\Gamma}_1 \\
\frac{\partial \hat{\Gamma}_1}{\partial t} - G_0 \hat{\Gamma}_1 - H_0 \hat{v}_1 \quad &= \quad -\frac{\partial \hat{\Gamma}_0}{\partial \tau} + G_1 \hat{\Gamma}_0 + H_1 \hat{v}_0 + K_1 \hat{\omega}_{y0} \\
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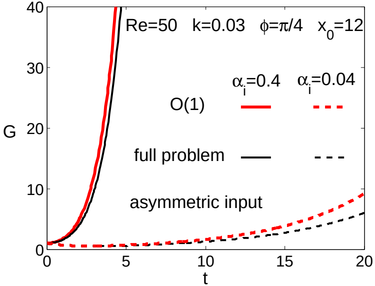
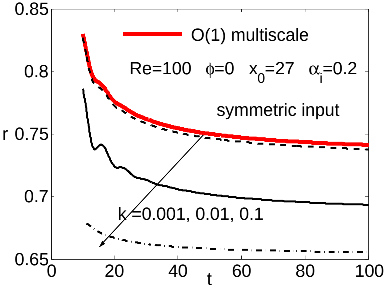
\end{aligned}
$$

where $G_1 = G_1(y, Y; x_0, \phi, \alpha_i, Re)$ and similarly for H_1, K_1, L_1 and M_1.

Effect of α_i and k





Scarsoglio, Tordella & Criminale, *Phys. Rev. E*, 2010.

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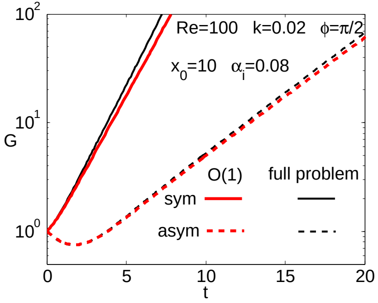
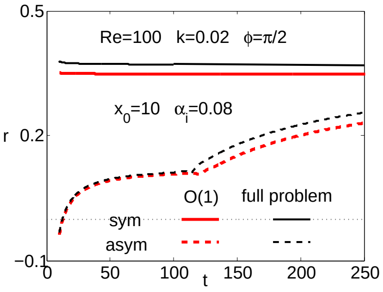
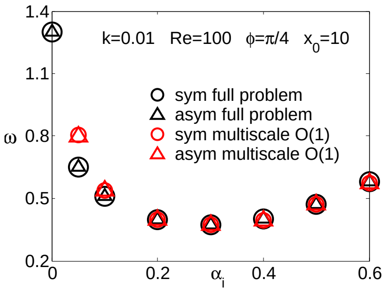
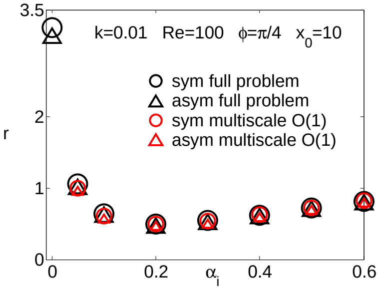




Asymptotic state

- Temporal asymptotic values of the angular frequency ω and the temporal growth rate r.





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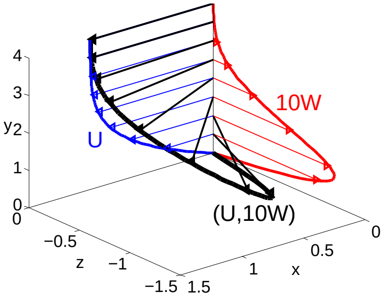

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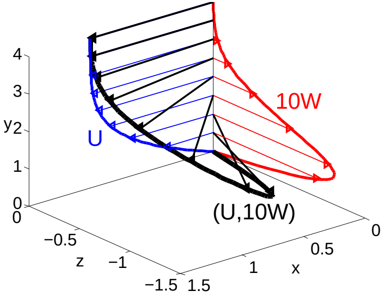

- Initial-value problem for compressible flows.